VOL. 24, No. 1, Sept.-Oct. 1950.

MATHEMATICS magazine

MATHEMATICS MAGAZINE

Formerly National Mathematics Magazine, founded by S. T. Sanders.

Address editorial correspondence to Glenn James, special papers to the editors of the departments for which they are intended, and general papers to some member of the Executive Committee.

Manuscripts should be typed on 8½" x 11" paper, double-spaced with 1" margins. We prefer that, in technical papers, the usual introduction be preceded by a *foreword* which states in simple terms what the paper is about.

The Mathematics Magazine is published at Pacoima, California by the managing editor, bi-monthly except July-August. Ordinary subscriptions are $3.00, sponsoring subscriptions $10.00, single copies 65¢. Reprints: 25 free to authors; 100 @ $1.25 per page with a minimum of $6.00. All reprints are bound.

Subscriptions and related correspondence should be sent to Inez James, 14068 Van Nuys Blvd., Pacoima, Calif.

Advertisers should contact Homer V. Craig, Univ. of Texas, Austin 12, Tex.

Entered as second-class matter March 23, 1948 at the Post Office, Pacoima, California under act of congress of March 8, 1876.

A complete list of sponsoring subscribers will appear in the Nov.-Dec. issue.

CONTENTS

HERMAN LYLE SMITH

Professor H. L. Smith of Louisiana State University died on June 13, 1950, without having been ill.

Professor Smith was born in Pittwood, Illinois, and spent most of his youth in Oregon. He attended the University of Oregon as an undergraduate but received the B.S., M.S., and Ph.D. degrees from the University of Chicago. The latter degree was conferred in 1926 with Professor E. H. Moore as Major Professor.

Professor Smith published some 25 papers in the National Mathematics Magazine, American Journal, Transactions, Annals of Mathematics and The Bulletin. His wide interest in mathematics is indicated by such titles as General Theory of Limits, co-authored by E. H. Moore (Vol. XLIX, American Journal), On the Existence of Stieltjes Integral (Vol. 27, Transactions), On functions of Closest Approximation (Vol. 31, The Bulletin), and On the Definition of the Sum of Two Factors (Vol. 6, Mathematics News Letter). This magazine was the forerunner of the National Mathematics Magazine. His most outstanding papers were the one co-authored by E. H. Moore and another with the same title which appeared in Vol. XII of the National Mathematics Magazine.

Professor Smith was an editor of The National Mathematics Magazine from the time it was founded and subsequently of the Mathematics Magazine. His combined knowledge of mathematics and Russian was brought into use in furnishing reviews for Mathematical Reviews. At the time of his death two persons were working with him as major professor toward the Ph.D. degree and several were writing masters theses under his supervision.

THE INVERSION OF THE LAPLACE TRANSFORMATION*

A. Erdélyi

1. In the literature on the Laplace transformation much attention has been devoted to *inversion formulae*, and a considerable number of these is known. In the theory, they often form the basis of representation theorems, that is, statements of necessary and sufficient conditions under which a function is a Laplace transform, or a Laplace transform of a function of a certain class; and in the applications inversion formulae are often used at the last stage of the solution of a problem by means of the Laplace transformation.

While many individual inversion formulae have been investigated, it seems that no general principle has developed that will yield all of them. Yet, there is such a principle, and it can be formulated briefly (if somewhat inaccurately) as follows. *If there is a singular integral whose kernel $N(u, t; k)$ can be interpreted as the result of a linear operation on e^{-su} so that $N = L_{k,t}[e^{-su}]$, then $L_{k,t}$ is an inversion operator for the Laplace transformation.*

The extension of this principle to the Laplace-Stieltjes transformation is immediate, and it is also clear that, with small changes, the principle applies to other linear functional transformations, notably integral transformations.

The principle is not new. In special cases it has been used repeatedly to *prove* individual inversion formulae. However, it does not appear to have been formulated in a general manner, nor does it seem to have been exploited for *discovering* inversion operators — except possibly for Stieltjes' discovery of the inversion by means of derivatives.

2. The Laplace transform of a function $\phi(t)$ is defined as

$$f(s) = \mathfrak{L}\left[\phi(t);\ s\right] = \int_0^\infty e^{-su}\phi(u)du \tag{1}$$

where $\int_0^\infty$ is to be interpreted as $\lim \int_0^R$ as $R \to \infty$. It is well known [8; Ch. II, § 1] that if the integral (1) converges for $s = s_0$, then it converges for all s for which Re $s >$ Re s_0, and for all such s

$$\mathfrak{L}\{\phi;\ s\} = (s - s_0)\mathfrak{L}\{\phi_1;\ s - s_0\} \tag{2}$$

where

$$\phi_1(t) = \int_0^t e^{-s_0 u}\phi(u)du, \tag{3}$$

*This article is based on a lecture given to the Peripatetic Seminar in Mathematics, at the University of Southern California, on June 5, 1950.

and the integral on the right hand side of (2) converges absolutely. Thus, it is sufficient to consider inversion formulae for absolutely convergent Laplace transforms of bounded continuous functions, and it is even permissible to assume that the determining function, $\phi(t)$, is absolutely continuous in any finite interval $(0, R)$.

For the Laplace-Stieltjes transform one has similarly

$$\int_0^\infty e^{-st} d\alpha(t) = (s - s_0)\mathfrak{L}\left\{\beta(t);\ s - s_0\right\} \tag{4}$$

where

$$\beta(t) = \int_0^t e^{-s_0 u} d\alpha(u) \tag{5}$$

and the Laplace integral on the right hand side of (4) converges absolutely.

Once an inversion formula has been established for absolutely convergent Laplace transforms, its extension to more general Laplace transforms and to Laplace-Stieltjes transforms follows comparatively well-established lines [cf. 8; ch. II § 7].

3. In order to illustrate the operation of the principle, we shall describe briefly the proofs of the two best known inversion formulae.

First, we discuss the *complex inversion formula* [8; Ch. II § 7]. Let $f(s) = \mathfrak{L}\left\{\phi;\ s\right\}$ be absolutely convergent for Re $s = c$ and hence uniformly convergent for Re $s \geq c$. Define

$$L_{k,t}[f(s)] = \frac{1}{2\pi i}\int_{c-ik}^{c+ik} e^{st} f(s)\,ds,\quad k > 0. \tag{6}$$

For $f(s)$ we substitute (1), invert the order of integration (by uniform convergence) and obtain

$$L_{k,t}[f(s)] = \int_0^\infty N(u,t;\ k)\phi(u)\,du \tag{7}$$

where

$$N(u,t;\ k) = L_{k,t}[e^{-su}] = e^{c(t-u)}\frac{\sin k(t-u)}{\pi(t-u)}. \tag{8}$$

If we assume that $\phi(u)$ is absolutely continuous in some open interval including $u = t$ (or under weaker conditions)

$$\lim_{k\to\infty}\int_0^\infty N(u,t;\ k)\phi(u)\,du = \phi(t) \tag{9}$$

is a consequence of Fourier's single integral theorem.

Secondly, we take the *inversion by means of derivatives* [8; Ch. VII § 6]. Here

$$(10)\qquad L_{k,t}[f(s)] = \frac{(-)^k}{k!}\left[\frac{k}{t}\right]^{k+1} f^{(k)}\left[\frac{k}{t}\right], \; k = 0,\ 1, \ldots .$$

It is well known that this operator can be applied to (1) under the integral sign [8; Ch. II § 5] so that we obtain (7) with

$$(11)\quad N = L_{k,t}[e^{-su}] = \frac{u^k}{k!}\left[\frac{k}{t}\right]^{k+1} e^{-ku/t},\; k = 0,\ 1,\ 2 \ldots .$$

With this N, the integral in (9) can be evaluated asymptotically, for large k, by Laplace's method [8; Ch. VII § 3] and it is seen that (9) holds.

4. The examples of section 3 lead to a more precise formulation of the general principle.

Let $L_{k,t}$ be a linear operator on a suitable class of analytic functions of s to the class of (numerical valued) functions of the positive real variable t. k may be a continuously varying parameter, or else restricted to a set with a limit point which may be taken at ∞. If the domain of $L_{k,t}$ includes all Laplace transforms (or all absolutely convergent Laplace transforms) and if $L_{k,t}$ can be applied to (1) under the integral sign, we have (7) with

$$(12)\qquad N(u,t;\ k) = L_{k,t}[e^{-su}].$$

Lastly, if N is a singular kernel so that

$$(13)\qquad \int_0^\infty N(u,t;\ k)\phi(u)du \to \phi(t),$$

in some sense, as $k \to \infty$, then $L_{k,t}$ is an inversion operator for the Laplace transformation, that is $L_{k,t}[f(s)] \to \phi(t)$ as $k \to \infty$.

It should be noted that (13) need not mean point-wise convergence. Other types of convergence are allowable: strong convergence (for instance convergence in mean) and weak convergence (for instance convergence to $\int_0^t \phi(u)du$) have occasionally been used.

Now, a large number of singular kernels is known. There is a general theory, and particular kernels may be constructed, for instance by the method of stationary phase (Fourier type) or by the method of steepest descent or Laplace's method (Gauss type). Once such a kernel N has been constructed one attempts to represent it in the form (12). If such a representation is possible and if the operator $L_{k,t}$ involved has the desired properties (suitable domain, linearity, etc.), then $L_{k,t}$ is an inversion operator.

At this point it is perhaps useful to recall that by Laplace's analysis [8; Ch. VII § 2]

$$(14) \quad N(u,t;\, k) = \left[\int_0^{\infty} A(v,t)e^{-kB(v,t)}dv\right]^{-1} A(u,t)e^{-kB(u,t)}$$

is a singular kernel in the sense of (13) provided (i) that $A(u,t)$ as a function of u is continuous in some neighbourhood of $u = t$ and integrable in any finite interval; (ii) $B(u,t)$ as a function of u is non-increasing when $0 < u < t$, non-decreasing when $u > t$, and C^2, in some neighbourhood of $u = t$ with $\partial^2 B/\partial u^2 > 0$ there; and (iii) the integral in (14) is absolutely convergent for all $k > K$. Moreover, this integral may be replaced (asymptotically) by

$$(15) \qquad \left[\frac{\pi}{2kB_{uu}(t,t)}\right]^{\frac{1}{2}} A(t,t)e^{-kB(t,t)}.$$

5. It remains to illustrate the general principle by a few examples.
5.1. With

$$(16) \qquad N = 1 - \exp\left[-e^{k(t-u)}\right]$$

it is fairly easy to see that

$$\lim_{k\to\infty}\int_0^{\infty} N\phi(u)du = \int_0^t \phi(u)du$$

Moreover, from (16),

$$N = \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{n!} e^{kn(t-u)}$$

and this leads to the definition of Phragmén's inversion operator [2; Ch. VII § 2]

$$L_{k,t}[f(s)] = \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{n!} e^{knt}f(kn).$$

5.2. As an illustration of a singular kernel which can be obtained from Fourier's kernel by a change of variables, I mention

$$\frac{\sin(k \log t/u)}{(ut)^{\frac{1}{2}}\pi \log t/u}$$

which leads to an inversion formula investigated by Paley-Wiener [5; § 13] and Doetsch [1].

Another such kernel is

$$\frac{\sin k(u^{\frac{1}{2}} - t^{\frac{1}{2}})}{2\pi u^{\frac{1}{2}}(u^{\frac{1}{2}} - t^{\frac{1}{2}})}$$

It probably leads to an inversion operator related to Kelvin's inversion formulae [4], but I have not investigated this case in detail.

5.3. Laplace's method is a fruitful source of singular kernels. As a first example we take in (14)

$$A = \left(\frac{u}{t}\right)^{\lambda\alpha-1}, \quad B = \left(\frac{u}{t}\right)^{\lambda} - \lambda \log \frac{u}{t}, \quad \lambda \neq 0 \text{ real}, \ \alpha \text{ real}.$$

Clearly the conditions on A and B are satisfied, and

$$(17) \qquad N = \frac{|\lambda| - k^{k+\alpha}}{\Gamma(k+\alpha)t}\left(\frac{u}{t}\right)^{\lambda k+\lambda\alpha-1} \exp\{-k\left(\frac{u}{t}\right)^{\lambda}\}.$$

For $\lambda = 1$, $\alpha = 0$ this is (11); for $0 < \lambda < 1$ Pollard [6] has proved that $L_{k,t}$ is an integral operator, and for $\lambda < 0$ the same result is very easy to prove. In particular, $\lambda = -1$, $\alpha = 0$ leads to Hirschman's inversion by means of Bessel functions [3], and $\lambda = \frac{1}{2}$, $\alpha = 1$ to an (unpublished) inversion by means of the functions of the parabolic cylinder, in particular by means of Hermite polynomials if k varies through integers.

5.4. Another case of (14) is

$$A = 1, \ B = (a+1)\log(u+at) - \log u, \ a > 0$$

$$(18) \quad N = \frac{\Gamma(ak+k)}{k!\Gamma(ak-1)} u^k (at)^{ak-1}(u+at)^{-ak-k} \quad k = 1, 2, \ldots$$

Here (12) leads to the definition

$$(19) \quad L_{k,t}[f(s)] = \frac{(-)^k (at)^{ak-1}}{k!\Gamma(ak-1)}\int_0^\infty e^{-ast} s^{ak+k-1} f^{(k)}(s)ds.$$

For $a = 1$ this is Boas-Widder's real inversion operator [8; Ch. VIII § 25].

5.5. As a last example of (14) we take

$$A = u^{-\nu-1}, \ B = \frac{u}{t} + \frac{t}{u},$$

Here [7; p. 182 (8)]

$$\int_0^\infty Ae^{-kB}du = 2t^{-\nu}K_\nu(2k)$$

where K_ν is the modified Bessel function of the third kind. The kernel is

$$N = [2K_\nu(2k)]^{-1} t^\nu u^{-\nu-1} \exp\{-k \left[\frac{u}{t} + \frac{t}{u}\right]\}. \tag{20}$$

By a well-known formula from Bessel function theory [7; p. 394 (4)] this is

$$[2tK_\nu(2k)]^{-1} k \int_0^\infty x^{\frac{1}{2}\nu} J_\nu(2kx^{\frac{1}{2}}) e^{-ku(x+1)/t} dx$$

and hence the inversion operator

$$L_{k,t}[f(s)] = [2tK_\nu(2k)]^{-1} k \int_0^\infty x^{\frac{1}{2}\nu} J_\nu(2kx^{\frac{1}{2}}) f(k(x+1)/t) dx \tag{21}$$

in which $K_\nu(2k)$ may be replaced by its asymptotic representation $\frac{1}{2}(\pi/k)^{\frac{1}{2}} e^{-2k}$. This inversion operator is essentially different from Hirschman's operator in that here the order of the Bessel function, ν, is fixed, while in Hirschman's work the order of the Bessel function is k.

The two particular cases $\nu = \pm\frac{1}{2}$ are especially interesting because then the Bessel functions reduce to elementary functions and we have the inversion operators

$$L_{k,t}[f(s)] = \frac{k}{\pi t} e^{2k} \int_0^\infty f(k(x+1)/t) \sin(2kx^{\frac{1}{2}}) dx \tag{22}$$

$$L_{k,t}[f(s)] = \frac{k}{\pi t} e^{2k} \int_6^\infty f(k(x+1)/t) x^{-\frac{1}{2}} \cos(2kx^{\frac{1}{2}}) dx. \tag{23}$$

The last three inversion operators reveal a connection between the inverse Laplace transformation on the one hand, and the Fourier sine and cosine transformation and (more generally) the Hankel transformation on the other hand.

There are many other interesting examples; but the few given here should suffice to show the scope of the method, and its power both to coordinate known inversions and to discover new ones.

References

1. G. Doetsch, Math. Zeitschr. 42 (1937) 263-286.
2. G. Doetsch, *Die Laplace Transformation* (1937).
3. I. I. Hirschman, Jr., Duke J. 15 (1948) 472-494.
4. Lord Kelvin, Math. and Phys. papers (1882) vol 1, p. 10-21. See also H. Bateman, Messenger of Math. 57 (1928) 145-154.
5. R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain* (1934).
6. H. Pollard, Bull. Am. Math. Soc. 52 (1946) 908-910.
7. G. N. Watson, *Theory of Bessel functions* (1922).
8. D. V. Widder, *The Laplace transform* (1941).

California Institute of Technology

PROOF OF A CONJECTURE OF R. C. AND E. F. BUCK[1]

Marlow Sholander

In a recent issue of this magazine in their paper "Equipartition of Convex Sets"[2], R. C. and E. F. Buck proved that no bounded plane convex set K can be partitioned into seven subsets of equal area by three non-concurrent lines. They proved further that if six of the seven subsets have equal area, these subsets are necessarily the six outer subsets. They conjectured that, in this case, the area of the (triangular) inner subset is at most one eighth the area of an outer subset. In this paper we prove the conjecture true, and show that this upper bound is attained only if K is a triangle which is partitioned by lines parallel to its sides.

In each of various convex sets K we consider, F, G, D, E, H, and I are points ordered positively on the boundary of K in such a way that $FACE$, $GBCH$, and $DBAI$ are non-concurrent lines dividing K into seven subsets [see Figure 2 of *ECS*]. Let a be the area of triangle ABC. Let b_1 (resp., b_2, b_3) be the area of the outer subset which has AB (resp., BC, CA) as a boundary segment. Let c_1 (resp., c_2, c_3) be the area of the remaining subset which has A (resp., B, C) as a corner. It is sufficient to consider sets K for which $b_1 = b_2 = b_3 \equiv b$ and for which $b \leqq \min c_i$, and to show $a/b \leqq 1/8$.

We hold (with *ECS*) it is obvious that a/b is bounded. It follows from Blaschke's Selection Theorem [see section 25 of Bonnesen and Fenchel, *Theorie der Konvexen Körper*, (reprinted in) New York, 1948] that the least upper bound is attained by some convex set K. We may verify (with *ECS*) that if K is a triangle and the three non-concurrent dividing lines are parallel to the sides of K, then $a/b \leqq 1/8$ and equality holds if and only if $b = c_1 = c_2 = c_3$. The proof may now be completed by showing that for all other convex sets K satisfying the given area relations, the ratio a/b is not maximum.

The following lemma is known [see *ECS*, Figure 2].

Lemma 1. If maximum a/b is attained for a non-triangle, it is attained for a triangle.

Proof: Let FG and ED (resp., DE and IH, HI and GF) meet at P (resp., Q, R). In forming the new convex set, triangle PQR, we have not changed a, not increased b_i, and not decreased c_i, for $i = 1, 2, 3$. The sides of PQR may now be translated outward to give a triangular set $P'Q'R'$ for which, using an obvious notation, $a = a'$, $b = b_1' = b_2' = b_3'$, and $\min_i c_i \leqq \min_i c_i'$. Thus $P'Q'R'$ is a set of the type

[1]This paper was written while the author was receiving support from the Office of Naval Research.

[2]Volume XXII, No. 4, pages 195-198 (1949). This paper is referred to, below, as *ECS*.

being considered and $a'/b' = a/b$.

We thus restrict our discussion (except for Lemma 4) to triangular sets PQR.

Lemma 2. If $b < \min_i c_i$, a/b is not maximum.

Proof: It is clear that in this case the sides of PQR may be translated a small distance inward in such a way that we preserve the equality of b_1, b_2, and b_3 as well as the inequality $b < \min_i c_i$. This transformation increases a/b by decreasing b.

Lemma 3. If a/b is maximum, $b = c_1 = c_2 = c_3$.

Proof: By Lemma 2, we may assume c_1, say, equals b. Assume, say, $c_2 > b$. We may fix lines DQ, QI, AB, BC, and CA and move line FG so that b_1 is fixed, c_1 increases, and c_2 decreases. In this way we may hold a/b fixed and obtain a triangle which, by Lemma 2, does not have maximum a/b.

The following lemma follows from Lemmas 2 and 3 and the proof of Lemma 1.

Lemma 4. If a/b is maximum, K is a triangle.

We continue by examining a triangle PQR with the property of Lemma 3. Since PQR may be made equilateral by an affine transformation and since this transformation preserves the ratio of areas, we may assume P, Q, and R are the points $(1, 0)$, $(0, \sqrt{3})$, and $(-1, 0)$ respectively. Then $a + 6b = \sqrt{3}$. Let $s = (a + 3b)/\sqrt{3}$ and $r = (a + b)/s\sqrt{3}$. It follows that $1/2 < s < 1$ and $1/3 < r = (5s - 2)/3s < 1$.

Let $\alpha = RG/RP$, $\alpha' = RH/RQ$, $\beta = PE/PQ$, $\beta' = PF/PR$, $\gamma = QI/QR$, and $\gamma' = QD/QP$.

Lemma 5. $\alpha\alpha' = \beta\beta' = \gamma\gamma' = s$ and $r = f(s, \alpha, \beta) = f(s, \beta, \gamma) = f(s, \gamma, \alpha)$ where $f(s, \alpha, \beta) = (s + \alpha\beta - \beta)^2/(s^2 - \beta^2 s + \alpha^2\beta^2)$.

Proof: To prove, say, $\alpha\alpha' = s$ we note $\alpha\alpha' = RG \cdot RH/RP \cdot RQ =$ area RGH/area $RPQ = (a + 3b)/\sqrt{3}$. To prove, say, $r = f(s, \alpha, \beta)$ we compute $F(1 - 2\beta', 0)$, $G(-1 + 2\alpha, 0)$, $E(1 - \beta, \beta\sqrt{3})$, $H(-1 + \alpha', \alpha'\sqrt{3})$, and the ordinate of C as

$$\sqrt{3}\, s\beta(s + \alpha\beta - \beta)/(s^2 - \beta^2 s + \alpha^2\beta^2).$$

Hence, we have $a + b =$ area $FGC = \sqrt{3}\, sf(s, \alpha, \beta)$.

Lemma 6. There exists a triangle for which a/b is maximum and for which α, β, and γ have a common value δ.

Proof: We consider the triangle of Lemma 5 and assume a/b is maximum. We note a/b depends only on s since $a = \sqrt{3}(2s - 1)$ and $b = (1 - s)/\sqrt{3}$. Our lemma is proved if there exists a δ, $0 < \delta < 1$, such that $r = (5s - 2)/3s = f(s, \delta, \delta)$. This equation is equivalent to the equation $g(s, \delta) = 0$ where $g(s, \delta) = 2\delta^4(1 - s) - 6s\delta^3 + (11s^2 + s)\delta^2 - 6s^2\delta + 2s^2(1 - s)$. We find

$$g(s, 1) = -2s^3 + 6s^2 - 7s + 2.$$

Since $\frac{d}{ds} g(s, 1) = -1 - 6(1 - s)^2 < 0$ and since $g(½, 1) = -¼$, $g(s, 1) <$

0. Since $g(s, 0) = 2s^2(1 - s) > 0$, our lemma is proved.

Lemma 7. If a/b is maximum and $\alpha = \beta = \gamma$, then AB (resp., BC, CA) is parallel to RP (resp., PQ, QR).

Proof: Assume these lines are not parallel. We may assume, as before, that PQR is equilateral. Since $\alpha = \beta = \gamma$, ABC is equilateral. Let the perpendicular bisector of AB (resp., BC, CA) meet RP (resp., PQ, QR) at C' (resp., A', B'). Let the line through C' (resp., A', B') parallel to AB (resp., BC, AC) be denoted by p_3 (resp., p_1, p_2). Let p_1 and p_2 (resp., p_1 and p_3, p_2 and p_3) meet at Q' (resp., P', R'). Consider triangle $P'Q'R'$. Clearly, using an obvious notation, $a' = a$, $b' < b$, and there is a number c' such that $c_1' = c_2' = c_3' = c'$. Since $a/b < a'/b'$ a contradiction results by showing $b' \leqq c'$. We note angle $A'Q'B'$ = angle $A'QB'$ and hence Q and Q' lie on a circle which has $A'B'$ as a chord. Since $A'Q' = Q'B'$, area $A'QB'$ < area $A'Q'B'$. Similarly we have the area inequalities $C'PA' < C'P'A'$ and $B'RC' < B'R'C'$. It follows that $a + 6b$ = area PQR < area $P'Q'R' = a' + 3b' + 3c'$ and that $b' < b < c'$.

By Lemmas 6 and 7 and elementary computations, we establish the desired theorem.

Theorem. We have $a/b \leqq 1/8$. Equality holds for a triangle in which $\alpha = \beta = \gamma = 5/7$ and $s = 25/49$.

Corollary. Equality holds only for the triangle of the preceding theorem (a triangle cut by lines parallel to its sides).

Proof: The problem of uniqueness is seen to depend on whether other solutions have been lost by the application of Lemma 6. There remains to be shown only the impossibility of having a triangle for which $s = 25/49$ and for which α, β, and γ are not equal. Assume such a triangle exists. It follows that $(5s - 2)/3s = f(s, \alpha, \beta) = f(s, \beta, \gamma) = f(s, \gamma, \alpha)$ where $s = 25/49$. From Lemma 5, we have $25/49 < \alpha, \beta, \gamma < 1$.

Let $\alpha = 5x/7$, $\beta = 5y/7$, and $\gamma = 5z/7$, Let $x^* = 1/x$, $y^* = 1/y$, and $z^* = 1/z$, The equalities in α, β, and γ reduce to the equalities $Q(x, y^*) = Q(y, z^*) = Q(z, x^*) = 0$ where $Q(x, y^*) =$

$$16x^2 + 50xy^* + 16y^{*2} - 70x - 70y^* + 58,$$

where $5/7 < x, y, z < 7/5$, and where, say, $x \neq y$.

Since $Q(x, y^*) = Q(y^*, x)$, we may assume $1 < x < 7/5$. [To assume $5/7 < x < 1$, merely interchanges α and α', β and β', γ and γ'. To assume $x = 1$ implies $y = y^* = 1$.] Since $y^* > 5/7$, we are concerned only with one branch of the hyperbola $Q(x, y^*) = 0$ and we have $1 > y^* > 3\sqrt{6}/10$. We note for use below that (since the axis of the hyperbola $Q(x, y^*) = 0$ is the line $y^* = x$) for these points, $x + y^* < 7/5 + 3\sqrt{6}/10 < 35/16$.

Denoting derivatives with respect to x by primes we have

$$y^{*\prime} = \frac{16x + 25y^* - 35}{35 - 25x - 16y^*}.$$

Let $w = xy^*$. We compute

$$w' = (x - y^*)\,\frac{16(x + y^*) - 35}{35 - 25x - 16y^*}.$$

From previous inequalities and the inequality $25x + 16y^* > 25 + 24\sqrt{6}/5 > 35$, we have $w' > 0$. Hence $w > 1$ and $x > y > 1$. Similarly, from $Q(y, x^*) = 0$ we have $y > z > 1$, and from $Q(z, x^*) = 0$ we have $z > x > 1$, We are led to the contradiction $x > x$.

Washington University

EXTRACT FROM PRELIMINARY ANNOUNCEMENT OF THE CLEVELAND MEETING OF THE AAAS

December 26-30, 1950.

The 117th Meeting of the American Association for the Advancement of Science, the annual meeting for the year 1950, will be a full-scale meeting – with programs in every principal field of science from astronomy and botany to, and including, zoology. All 17 of the Association's sections and subsections, and more than 40 participating societies and organizations, are completing plans for an aggregate of more than 200 sessions.

There will be a considerable number of outstanding symposia. The list of special sessions includes the AAAS Presidential Address, and addresses sponsored by the Academy Conference, the National Geographic Society, the Scientific Research Society of America, the Society of the Sigma Xi, and the United Chapters of Phi Beta Kappa.

The American Museum of Atomic Energy will be a part of the *ANNUAL SCIENCE EXPOSITION* at the Cleveland Meeting.

COLLEGIATE ARTICLES

Graduate training not required for Reading

THE LOGARITHMIC FUNCTION IS UNIQUE*

Joseph Milkman

The operation of the slide rule for multiplication and division depends on the property of the function $\log x$, that $\log a + \log b = \log ab$. A slide rule with different scales could be made if there were three continuous functions, f, g, and h such that

$$f(x) + g(y) = h(xy). \tag{1}$$

The object of this paper is to show that if f, g, and h are continuous functions of real variables satisfying the functional equation (1) then $f(x) = k \ln c_1 x$, $g(y) = k \ln c_2 y$, $h(xy) = k \ln c_1 c_2 xy$. We shall also establish that the logarithmic function is the only solution of (1) if we broaden our domain of admissible functions. The paper will also show how one can easily obtain the elementary properties of the logarithmic function from the functional equation

$$f(x) + f(y) = f(xy). \tag{2}$$

Throughout this paper when we speak of a function, we shall assume it is defined for all positive real numbers.

Theorem I. If $f(x)$ satisfies (2) and n and m are positive integers,

$$f(x^n) = n\, f(x) \tag{3}$$

$$f(x^{m/n}) = \frac{m}{n} f(x) \tag{4}$$

$$f(1) = 0 \tag{5}$$

$$f(x^{-m/n}) = -\frac{m}{n} f(x) \tag{6}$$

$$f\left[\frac{x}{y}\right] = f(x) - f(y) \tag{7}$$

Proof: $f(x) = 1 \cdot f(x)$ and if $f(x^n) = nf(x)$, then $f(x^{n+1}) = f(x) + f(x^n) = (n + 1)f(x)$. Therefore (3) is true for all n by the axiom of mathematical induction.

$$f(x) - f(x^{n/n}) = nf(x^{1/n}) \text{ or}$$

$$f(x^{1/n}) = \frac{1}{n} f(x), \text{ but}$$

*Presented to the Mathematical Association of America on Dec. 4, 1948.

$$f(x^{m/n}) = f[(x^{1/n})^m] = mf(x^{1/n}) = \frac{m}{n} f(x)$$

which is (4). $f(1) + f(1) = f(1)$, hence (5). (6) follows from $f(x^{m/n}) + f(x^{-m/n}) = f(1) = 0$; and (7) from $f\left(\frac{x}{y}\right) = f(x) + f(y^{-1})$.

Theorem II**. If $f(x)$ satisfies

(2) $$f(x) + f(y) = f(xy)$$ and is

H_1: continuous or
H_2: bounded in a closed interval $0 < a \leq x \leq b$ or
H_3: continuous at one pt $\cdot\, x_0$, then

$$f(x) \equiv k \ln x$$

where k is a constant.

Proof: Part I. By Theorem I $f(x^r) = rf(x)$ where r is any rational number. Suppose H_1 holds, then $f(e^x)$ is a continuous function of x. If $\bar{x}$ is an irrational number, there exists a sequence of rational numbers $\{x_n\}$ having $\bar{x}$ as a limit. Now $f(e^{x_n}) = x_n f(e)$ therefore

$$f(e^{\bar{x}}) = \lim_{x_n \to \bar{x}} f(e^{x_n}) = \lim_{x_n \to \bar{x}} x_n f(e) = \bar{x}\, f(e).$$

Therefore $f(e^r) = r\, f(e)$ for any real number r. Let y be any positive number, then $f(y) = f(e^{\ln y}) = (\ln y)f(e)$ is called k our theorem follows from H_1.

Part II. (2) and H_2 imply H_1.

Case I: $0 < a < 1 < b$.

Let M be an upper bound of $f(y)$ for $a \leq y \leq b$. We note that $a \leq (1 + x)^n < b$ if $\sqrt[n]{a} - 1 < x < \sqrt[n]{b} - 1$. Given any $\epsilon > 0$, let $n > M/\epsilon$ and let δ = the smaller of the quantities $1 - \sqrt[n]{a}$ and $\sqrt[n]{b} - 1$ then if $|x| < \delta$,

$$|f(1 + x)| = \frac{1}{n}|f[(1 + x)^n] \leq \frac{M}{n} < \frac{M}{M/\epsilon} = \epsilon,$$

or $$\lim_{x \to 0} |f(1 + x)| = 0.$$

But $$|f(x + h) - f(x)| = \left|f\left(\frac{x + h}{x}\right)\right| = \left|f\left(1 + \frac{h}{x}\right)\right|$$

therefore $$\lim_{h \to 0} |f(x + h) - f(x)| = 0,$$

i.e., $f(x)$ is continuous for all $x > 0$.

Case II: $0 < a \leq x \leq b \leq 1$.

**Cauchy, in his ''Cours de L'Ecole Royale Polytechnique'' Part I Analyse Algebrique 1821, Chap. V proved that the only continuous solution of $f(x + y) = f(x) + f(y)$ is $f(x) = ax$ and then showed that this implies that the only continuous solution of $f(x) + f(y) = f(xy)$ is $f(x) = k \log x$.

Let $$\lambda = \frac{1}{2a} + \frac{1}{2b}$$

then $$0 < \lambda a < 1 < \lambda b.$$

If $z = \lambda x$, $f(z) = f(\lambda x) = f(x) + f(\lambda)$ is bounded for $\lambda a \leq z \leq \lambda b$, since $f(x)$ is bounded for $a \leq x \leq b$. Therefore $f(z)$ is continuous for all $z > 0$ by Case I.

Case III. $1 \leq a \leq x \leq b$.

$$0 < \frac{1}{b} \leq \frac{1}{x} \leq \frac{1}{a} \leq 1$$

$\left| f\left(\frac{1}{x}\right) \right| = |f(x)|$ which is bounded. Hence $f(x)$ is continuous by Case II.

Part III. If $f(x)$ is continuous at x_0, then given $\epsilon > 0$,

$$|f(x) - f(x_0)| < \epsilon \text{ for } |x - x_0| < \delta,$$

$$|f(x)| \leq |f(x) - f(x_0)| + |f(x_0)| < \epsilon + f(x_0),$$

therefore (2) and H_3 implies H_2. The following corollaries are evident:

Corollary I. If $f(x)$ satisfies (2) and has one pt. of discontinuity, it is discontinuous everywhere.

Corollary II. If $f(x)$ satisfies (2) and is unbounded in any interval, it is unbounded in every interval.

Theorem III. If $f(x)$, $g(x)$, and $h(x)$ are functions satisfying

(1.) $$f(x) + g(y) = h(xy)$$

and $F(x)$ is the solution of $F(x) + F(y) = F(xy)$ (2), then

$$f(x) = F(x) + g(1)$$

$$f(x) = F(x) + g(1) + b$$

$$h(x) = F(x) + 2g(1) + b.$$

Proof:

$$f(x) + g(y) = h(xy)$$

$$f(y) + g(x) = h(yx) = h(xy)$$

$$f(x) - g(x) = f(y) - g(y).$$

Select a value of y and vary x and $f(x) - g(x) = b$ a constant or $f(x) = g(x) + b$ substituting in (1) gives

$$g(x) + b + g(y) = h(xy)$$

for $y = 1$ $$g(x) + b + g(1) = h(x),$$

so that $$g(x) + b + g(y) = g(xy) + b + g(1)$$

$$g(x) - g(1) + g(y) - g(1) = g(xy) - g(1).$$

Let $F(x) = g(x) - 1$ then $F(x)$ is any solution of (2).

Corollary: If $f(x)$, $g(x)$ and $h(x)$ are 3 functions satisfying the functional equation $f(x) + g(y) = h(xy)$ and $g(x)$ satisfies H_1 or H_2 or H_3 of Theorem II, then

$$f(x) = k \ln c_1 x$$

$$g(x) = k \ln c_2 x$$

and $$h(x) = k \ln c_1 c_2 x.$$

Theorems II and III may be generalized to n variables as follows:

Theorem II. If $x_1, \ldots, x_n$ are n independent variables and $f(u)$ is a function satisfying H_1 or H_2 or H_3 and (2) $f(x_1) + f(x_2) + \ldots + f(x_n) = f(x_1 x_2 x_3 \ldots x_n)$ then

$$f(u) = k \ln u.$$

Proof: $$f(1) + f(1) + \ldots + f(1) = f(1)$$

therefore $$f(1) = 0.$$

$$f(x_1) + f(x_2) + (n - 2)f(1) = f(x_1 x_2)$$

$$f(x_1) + f(x_2) = f(x_1 x_2).$$

Conversely if $f(x) + f(y) = f(xy)$, (2) follows by induction. Therefore Theorem II[1] follows at once from Theorem II.

Theorem III . If $f_i(u)$, $i = 1, 2 \ldots, n + 1$ are n functions satisfying the functional equation (1) $f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n) = f_{n+1}(x_1 x_2 \ldots x_n)$ and $f(x)$ is the solution of $f(x) + f(y) = f(xy)$ then

$$f_i(u) = f(u) + f_i(1).$$

Proof: $f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n) = f_1(x_i) + f_2(x_2) + \ldots + f_{i-1}(x_{i-1}) + f_i(x_1) + f_{i+1}(x_{i+1}) + \ldots + f_n(x_n)$, therefore $f_i(x_i) - f_1(x_i) = f_i(x_1) - f_1(x_1) = a_i$ for $i = 2, 3, \ldots, n$. $f_i(x_i) = f_1(x_i) + a_i$ substituting in (1[1]) and letting $x_i = 1$ for $i = 2, 3, \ldots, n$, we get $f_1(x_1) + (n - 1)f_1(1) + a_2 + a_3 + \ldots + a_n = f_{n+1}(x_1)$ therefore $f_1(x_1) + f_1(x_2) + \ldots + f_1(x_n) + a_2 + a_3 + \ldots + a_n = f_1(x_1 x_2 \ldots x_n) + (n-1)f_1(1) + a_2 + a_3 + \ldots + a_n$ let $F(x) = f_1(x) - f_1(1)$ and just as in proof of Theorem II[1], $F(x)$ is solution of $F(x) + F(y) = F(xy)$. Therefore, the theorem is established for $f_1(x)$. Similarly, it follows for $f_j(x)$.

I wish to express my appreciation to Professor S. S. Saslaw of the U.S. Naval Academy for suggesting the method of approximating real numbers by rationals used in the proof of Part I of Theorem II and to Professor Hyers, Editor of Mathematics Magazine for a number of suggestions.

U.S. Naval Academy, Annapolis, Md.

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

GUIDE-LINES

W. R. Ransom

A badly drawn curve is as inexcusable as a carelessly done computation; better results can be obtained by using a few guide-*lines* than by connecting a large number of calculated *points*. Figure 1 shows guide-lines for the graphs of sines, tangents, secants, powers, and exponentials. For the trigonometric curves the best guides are the tangents at points of inflexion, circles at points of extreme curvature, and asymptotes. For power and exponential curves, simple rules for sub-tangents furnish guides.

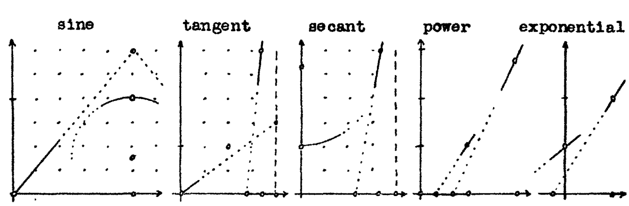

Fig. 1

For the trigonometric curves, $y = a \sin bx$, $y = a \tan bx$, etc., the scales are numbered to indicate the values of x and y, but the curve is drawn upon squared paper (corners of the squares are represented by dots in Fig. 1) and the measurements by which guide-lines are obtained are all made by counting squares, and not by reference to the scales for values of x and y. W is the number of squares representing a quadrant, that is to say W squares from the origin is marked on the scale with the value of x for which bx is a right angle. H is the number of squares that represents $y = a$, the coefficient of the sine, tangent, or secant in the equation.

For the fundamental theory, we note that if W squares represents a right angle, $N°$ is represented by $NW/90$ squares, and if N is so small that we cannot distinguish graphically between the arc and the perpendiculars (see Fig. 2), the value of sin $N°$ or tan $N°$ may be taken as $N\pi/180$. For the cosine and secant, we obtain approximations based upon this and upon the identities

$$(1 - \tfrac{1}{2}\sin^2 N°)^2 = 1 - \sin^2 N° + \tfrac{1}{4}\sin^4 N°$$

and

$$(1 - \tfrac{1}{2}\sin^2 N°)(1 + \tfrac{1}{2}\sin^2 N°) = 1 - \tfrac{1}{4}\sin^4 N°$$

For small values of $N°$, the term $\tfrac{1}{4}\sin^4 N°$ is so very small that we can neglect it and use the approximations

$$1 - \tfrac{1}{2}\sin^2 N° = \sqrt{1 - \sin^2 N°} = \cos N°$$

and

$$1 + \tfrac{1}{2}\sin^2 N° = 1/(1 - \tfrac{1}{2}\sin^2 N°) = \sec N°$$

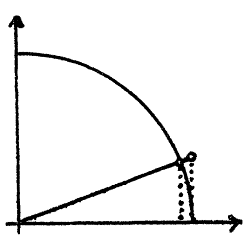

Fig. 2

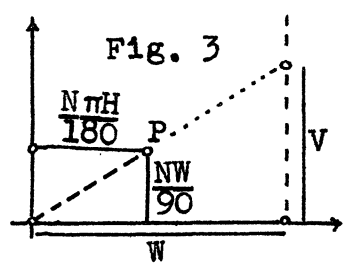

Fig. 3

When we plot the point P, (see Fig. 3) for sin $N°$ or tan $N°$, we lay off $NW/90$ squares horizontally and $N\pi H/180$ squares vertically. The line through the initial point and P cuts the vertical at the end of the quadrant at a certain height, V squares, and we have the proportion $V : W = N\pi H/180 : NW/90$, which gives $V = \pi H/2$. Since only a short piece at the left end of this guide-line is used, the more convenient value, $V = 3H/2$, can be used for drawing the initial tangent.

To find a circle for guide-line to a sinusoid we use the cosine rather than the sine, and employ the fact that PE (see Fig. 4) is a mean proportional between EF and the diameter of the desired circle. For a small enough angle, $N°$, PE will be indistinguishable from PF, the distance that represents $N°$; so we may take $PE = NW/90$. For EF we may take $H(1 - \cos N°)$, which by the approximation above is $\tfrac{1}{2}H\sin^2 N°$, or $\frac{1}{2} H(N\pi/180)^2$. The mean proportional relation then gives the equation

$$(NW/90)^2 = \text{Diameter} \cdot \tfrac{1}{2} H(N\pi/180)^2$$

whence Diameter $= 8W^2/\pi^2 H$. Since $\pi^2 = 9.870$, there will be no sensible error in taking $\pi^2 = 10$, and we get Radius $= 2W^2/5H$. In the diagram in Fig. 1, $W = 5$ squares, $H = 4$ squares, and the radius is $2\frac{1}{2}$ squares.

A similar argument shows that for the graph of the secant, EF' (Fig. 4) has the same approximate formula as EF, and the radius for the secant comes out $2W^2 \cdot 5H$ likewise.

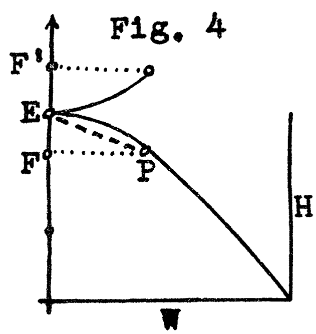

Fig. 4

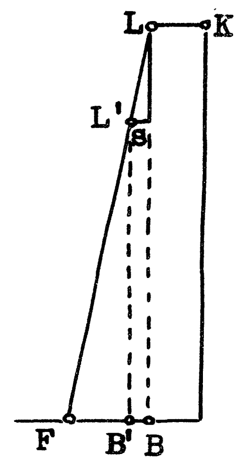

Fig. 5

Suppose L (see Fig. 5) is the last point plotted as the graph of the tangent or secant nears the asymptote, and is $M°$ from the end of the quadrant. Then $LK = D$ squares represents $M°$, and so D squares $= WM/90$ squares, and $M = (D/W)90$. At $(90 - M)°$ the tangent and secant are nearly equal in value, and the latter is $1/\sin M°$ and so the height of the point L is $(180/\pi M)H$ squares, or $2WH/\pi D$ squares. It follows that if L' is a point so close to L that its distance from the asymptote is $D + S$ squares where S is very much smaller than D, we have the height of L' equal to $2WH/\pi(D + S)$ squares. Subtracting, we find that the drop from L to L' is

$$\frac{2WH}{\pi}\left[\frac{1}{D} - \frac{1}{D+S}\right] \text{ or } \frac{2WH}{\pi D} \cdot \frac{S}{D+S}$$

Then by similar triangles we get

$$\text{drop} : S = LB : BF, \text{ or } \frac{2WH}{\pi D}\,\frac{S}{D+S} : S = \frac{2WH}{\pi D} : BF$$

whence $BF = D + S$, and S was taken so small that we may take AF as $2D$ without sensible error.

Another approximation formula is useful for deriving rules for drawing tangents at known points on the curves $y = kx^n$ and $y = kM^x$. The plan is to lay off a subtangent at the foot of the ordinate and connect its other end with the point on the curve. If U is near enough to unity, the interpolation formula

$$\log_{10} U = .4343(U - 1)$$

is readily checked by means of a seven-place table.

Let (a, b) and (a', b') be points on the curve so near to each other that a line accurately determined by these two points will be indistinguishable for graphical purposes from the tangent at either of the points. The subtangent is the ordinate divided by the slope of this line.

To find this slope in the case of $y = kx^n$, we obtain by substitution and division, $b'/b = (a'/a)^n$, whence

$$\log(b'/b) = n \log(a'/a),$$

or

$$.4343(b' - b)/b = n \cdot .4343(a' - a)/a$$

It follows that the slope is given by $(b' - b)/(a' - a) = n \cdot b/a$ and the subtangent by

$$b \div nb/a = \text{abscissa} \div \text{exponent}.$$

Similarly, in the case of $y = kM^x$, we get $b'/b = M^{(a'-a)}$ whence

$$\log(b'/b) = (a' - a)\log M, \text{ or } .4343(b' - b)/b = (a' - a)\log M$$

and the slope is given by

$$(b' - b)/(a' - a) = b(\log M)/.4343 = b \log_e M$$

and the subtangent by

$$b \div (b \log_e M) = \text{the constant, } 1/(\log_e M)$$

In the diagrams in Fig. 1, the curve $y = \sqrt{x^3}$, has abscissa divided by 3/2 for its subtangent, and the curve $y = 2^x$ has the constant subtangent, $1/(\log_e 2) = 1.44+$. Such subtangents are measured in units of of the horizontal scale.

Tufts College

NON-EUCLIDEAN GEOMETRY

Herbert Busemann

1. *Introduction.* The root of non-euclidean geometry is the problem whether our ordinary geometry[1] is the only possible one.

For centuries mathematicians were convinced that the answer must be positive and tried to prove this. Since all postulates of Euclid seemed self evident, except for the *parallel axiom*,[2] the efforts were directed towards establishing this axiom as a consequence of the others. *The failure of all these efforts led Gauss, Bolay and Lobachewsky during the first third of the 19th century to the conclusion that the parallel axiom is not a logical necessity but that consistent so-called non-euclidean geometries exist, in which this axiom does not hold.*

Since then mathematical research has enlarged our concepts regarding geometry and space to such an extent that the non-euclidean geometries appear now as very special examples of general geometric structures which are not euclidean (so that the terms non-euclidean and not euclidean are not synonymous). From the modern point of view the euclidean and the non-euclidean geometries resemble each other very closely. They can be derived jointly from a principle which was discovered by the physicist Helmholtz.[3]

In an implicit form the principle appears in Euclid's ''proof'' of the first congruence theorem: the triangles ABC and $A'B'C'$ are congruent if $AB = A'B'$, $AC = A'C'$ and $\angle BAC = \angle B'A'C'$. Euclid *moves* $\angle B'A'C'$ until it falls on $\angle BAC$. Then B' must fall on B and C' on C, etc. Clearly this argument is not the same type of proof as Euclid's other deductions. In fact the presupposed mobility of figures is equivalent to the congruence theorem and should therefore have been stated as an axiom. Helmholtz emphasized that *the mobility of rigid objects, the fact that we recognize an object independently of its position in space, is one of our basic physical experiences.* He therefore put, and partially solved, the mathematical problem of finding those spaces or geometries in which figures can be moved freely.

The answer to the problem is that the euclidean and the non-euclidean geometries have that property and that no other geometry does. A complete proof of this fundamental fact is beyond the scope of the present article, but one typical argument will indicate the procedure by which the result is obtained.

2. *Spherical geometry.* The reader is familiar with one non-euclidean geometry without realizing it, namely with the geometry on a sphere. Let Σ be (the surface of) a sphere with radius ρ and center O in the ordinary or euclidean space. Distances on Σ will be measured not in the space but along Σ. Airplane travel has made everyone familiar with the fact that shortest connections on Σ are arcs of great circles. (A great circle is a circle in which a plane through O intersects Σ.)

From the point of view of spherical geometry going along a great circle means choosing the most direct or straight route from one point to another. The great circles play therefore on Σ the role of straight lines and we shall frequently call them so.

If A,B are points on Σ and α is the radian measure of $\angle AOB$ then the length of the great circle arc (or the segment) from A to B is $\rho\alpha$, the spherical distance of A and B.

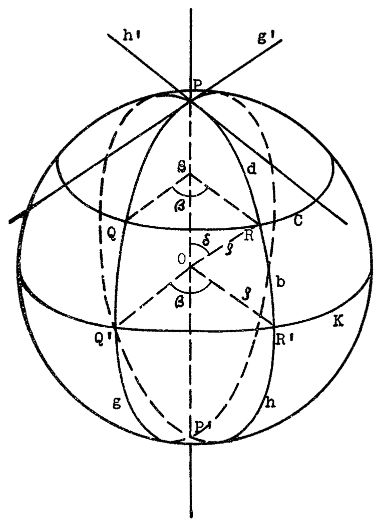

Figure 1

Let the two great circles g and h intersect at P and P'. These points are the intersections with Σ of the line which is common to the planes in which g and h lie. P' and P are diametrically opposite points of Σ or antipodes.

The circle C with radius d and center P on Σ is by definition the locus of those points which have spherical distance d from P. Let $R(Q)$ be one of the two intersections of $h(g)$ with C. If $d = \rho . \delta$ then $\delta = \angle QOP = \angle ROP$. Now C is also the euclidean circle whose radius is the perpendicular distance RS of R from the line PP'. The figure shows that $RS/RO = RS/\rho = \sin\delta$ or $RS = \rho \sin\delta$ so that C has length

$$L = 2\pi\,\rho \sin\delta = 2\pi\,\rho \sin\frac{d}{\rho} \tag{1}$$

This is the first of a series of non-euclidean formulas to be derived here. It solves the problem: given the radius d of a circle, find its length. The euclidean answer is $2\pi d$, the spherical answer is $2\pi\rho \sin d/\rho$. The constant ρ is typical for the specific spherical geometry under consideration. *It is a length and can be determined by purely geometric means* (for instance, the distance of any two antipodes is $\pi\rho$), *whereas in euclidean geometry no definite length* *can be determined by geometry alone.* We must use physical aids like the standard meter at Paris. The existence of absolute lengths is one of the main differences between the euclidean and the two non-euclidean geometries.

The relation $\sin x/x \to 1$ for $x \to 0$, which is familiar from calculus, and (1) show that

$$2\pi\ \rho \sin \frac{d}{\rho}/2\pi d = \sin \frac{d}{\rho}/\frac{d}{\rho} \to 1 \text{ for } d \to 0.$$

This means that for d which are small compared to ρ the spherical length of a circle is approximately equal to the euclidean length $2\pi d$. We recognize here a special case of a procedure which we use all the time, namely, *the application of euclidean geometry to small domains on the earth.*

Among the circles about P there is a spherical straight line k, the equator for P and P' as poles, which corresponds to the value $\delta = \pi/2$. Viewed from k the previously considered circle C appears as the locus of those points which have the same distance $b = QQ' = RR' = \rho(\pi/2 - \delta)$ from k (see Figure 1). Hence, *in spherical geometry the locus of the points which are equidistant to a straight line is not a straight line.* Also the lengths of the corresponding arcs $Q'R'$ and QR of k and C are not equal. for if β is the radian measure of the angle $\angle Q'OR'$ and $\angle QSR$ then $Q'R'$ has length $s = \rho\beta$ and the arc QR of C has length $t = \beta\rho \sin \delta$ so that

(2) $$t = s \sin \delta = s \cos(\pi/2 - \delta) = s \cos b/\rho$$

That t is proportional to s was obvious beforehand, (2) determines the factor of proportionality.

Hitherto only distances or lengths of curves were considered. In euclidean geometry angle and area are two other fundamental concepts. The same goes for spherical geometry. There is no question regarding angles, because locally, or close to a fixed point P, the spherical geometry is almost euclidean, so that angles with vertices at P can be measured in terms of the local euclidean geometry. This amounts to the following: instead of operating on Σ we operate on the tangent plane π to Σ at P. The two great circles g and h have tangents g' and h' at P which lie in π. As measure of the angle between g and h we take the measure of the euclidean angle of g' and h' in π (any unit can be used. Here we always use radian measure). Then the previously used β is also the measure of one of the angles formed by g and h, and we find

(3) $$C = \beta\rho \sin \frac{d}{\rho}$$

as spherical formula for the length of a circular arc with radius d which subtends the spherical angle β at its center P.

Everyone knows the importance of *Pythagoras' Theorem* in euclidean geometry. Unfortunately it is frequently formulated as a relation between areas. The reader, who remembers the applications of the theorem (for instance in trigonometry) will realize that its importance is due to the fact that *it expresses one side of a right triangle in terms of the other two sides.* The corresponding problem is just as important for spherical geometry. To solve it consider on Σ the triangle ABC with a right angle at C. Denote the (spherical) lengths of the sides AB, BC, CA by c, a, b respectively (all sides are assumed to be shortest connections of their endpoints). Let the tangent plane π of Σ at A (see Figure 2) intersect the line OB in B' and OC in C'. The plane is perpendicular to the radius OA and therefore to the plane OAC. The plane OBC is also perpendicular to OAC because the angle at

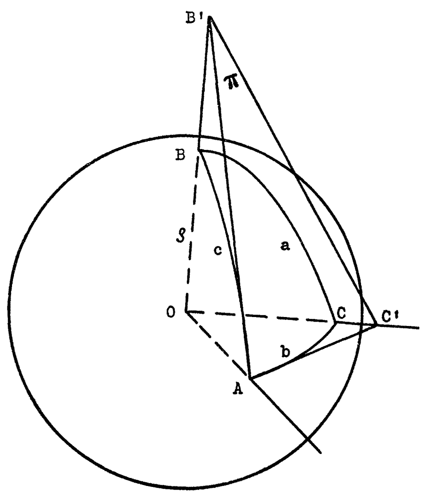

Figure 2

C is a right angle. Therefore the line $B'C'$ is perpendicular to the plane OAC, hence

$$<B'C'A' = <B'C'O = <B'AO = <C'AO = \pi/2$$

Since $<AOB' = c/\rho$, $<B'OC' = a/\rho$, $<C'OA = b/\rho$ we find

$$OA/OB' = \cos c/\rho, \quad OC'/OB' = \cos a/\rho, \quad OA/OC' = \cos b/\rho$$

and because of $OA/OB' = (OC'/OB')\cdot(OA/OC')$

(4) $\cos c/\rho = \cos a/\rho \cdot \cos b/\rho$ (*Spherical Theorem of Pythagoras*)

Those who are familiar with the expansion of $\cos x$ see that

$$1-\frac{c^2}{2\rho^2}+\dots=(1-\frac{a^2}{2\rho^2}+\dots)(1-\frac{b^2}{2\rho^2}+\dots)=1-\frac{a^2}{2\rho^2}-\frac{b^2}{2\rho^2}-\frac{a^2b^2}{4\rho^4}+\dots$$

or

$$c^2 = a^2 + b^2 + \ldots,$$

where the omitted terms contain at least ρ^2 in the denominator. This confirms again that the *euclidean geometry holds for triangles whose sides are small compared to ρ.*

The angle at A of the spherical triangle ABC equals (by our agreement) the euclidean angle $\angle B'AC'$. Therefore

$$\text{(5)} \qquad \cos A = \frac{AC'}{AB'} = \frac{AC'/OA}{AB'/OA} = \frac{\tan b/\rho}{\tan c/\rho} = \frac{\sin b/\rho}{\sin c/\rho}\cdot\frac{\cos c/\rho}{\cos c/\rho}$$

By (4) the last factor on the right equals $\cos a/\rho$. To evaluate the preceding factor observe that

$$\text{(6)} \qquad \sin A = \frac{B'C'}{AB'} = \frac{B'C'/B'O}{AB'/B'O} = \frac{\sin a/\rho}{\sin c/\rho}$$

Since A and B play the same role in the triangle ABC, the relation

$$\text{(7)} \qquad \sin B = \frac{\sin b/\rho}{\sin c/\rho}$$

must also hold. (5) and (7) yield

$$\text{(8)} \qquad \cos A = \sin B\cdot\cos a/\rho$$

Contrary to all our euclidean experiences (8) reveals that *in spherical geometry the angles of a triangle determine its sides.* A theory of similar figures does therefore not exist. *There is no such concept as shape independent of size.* This is, of course, closely related to the previously discussed existence of an absolute length.

Finally we turn our attention to area. A two-gon on Σ is a figure bounded by two semi great circles h, k with diametrically opposite points P, P' as vertices. Obviously the area A of a two-gon is proportional to the angle α at its vertices:

$$A = \omega\cdot\alpha$$

where ω is the factor of proportionality. Consider a triangle PQR, where Q and R lie on h and k. Let β and γ be the angles at Q and R of the spherical triangle PQR and Q', R' the antipodes to Q and R. Clearly any triangle is congruent to the triangle formed by its antipodes and has therefore the same area so that, for instance, $PQR = P'Q'R'$ and $PQR = P'Q'R$.

Now $PQR + PQR'$ is a two-gon with angle γ, hence

$$\text{(9)} \qquad PQR + PQR' = \omega\gamma$$

Similarly

$$\text{(10)} \qquad PQR + PRQ' = \omega.\beta$$

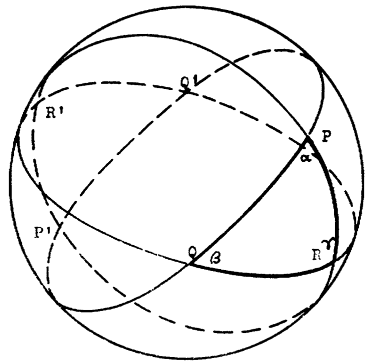

Figure 3

But $PQR' = P'Q'R$ and $PRQ' + P'Q'R$ is a two-gon with angle $\pi - \alpha$ so that

$$(11) \qquad PQR' + PRQ' = P'Q'R + PRQ' = \omega(\pi - \alpha)$$

Adding (9) and (10) and subtracting (11) yields

$$(12) \qquad 2\, PQR = \omega(\alpha + \beta + \gamma - \pi)$$

The area of a triangle is therefore proportional to the excess of the angle sum over π. This confirms that the size of a triangle is determined by its angles. The constant ω depends on the unit of area used. The usual agreement is to make the area of a spherical triangle as object of spherical geometry equal to the area of the same triangle as curved surface in euclidean space. This amounts to putting $\omega = 2\rho^2$.

A noteworthy consequence of (12) is that *the sum of the angles of a spherical triangle is always greater than π.*

Spherical geometry was, of course, well known prior to 1800, in fact, it was known to Ptolemy and the Arabs. Why then, did the discovery of non-euclidean geometry come so late? Here is the reason. Spherical geometry does not satisfy some of the euclidean axioms. Two straight lines (or great circles) intersect twice. There is however a geometry, closely related to the spherical, the so-called elliptic geometry, in which straight lines intersect only once. But in elliptic geometry only figures which are not too large can be moved freely. Moreover, neither in spherical nor in elliptic geometry can we continue walking straight ahead without eventually retracing our steps. Euclid assumes that this does not happen. The pioneers looked only for geometries in which the straight lines have the euclidean property of infinite extension in both directions. Thus they discovered the other non-euclidean geometry which is called hyperbolic. Since this geometry cannot be interpreted as geometry on a surface in ordinary space it is much more recondite.

After hyperbolic geometry had been fully developed its formulas proved amazingly similar to spherical geometry and unification became natural.

3. *The hyperbolic functions.* The analogy between spherical and hyperbolic geometries becomes apparent when the so-called hyperbolic trigonometric functions are introduced.

The hyperbolic sine and cosine of x are defined by the formulas:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \tag{13}$$

Clearly

$$\frac{d \sinh x}{dx} = \cosh x \text{ and } \frac{d \cosh x}{dx} = \sinh x \tag{14}$$

Since $e^x > 0$ for all x, the function $\cosh x$ is positive, therefore $\sinh x$ increases monotonically over the whole x-range. Because $\sinh 0 = 0$ it follows that

$$\sinh x < 0 \text{ for } x < 0 \text{ and } \sinh x > 0 \text{ for } x > 0$$

Therefore (14) and the elementary theory of maxima and minima yield that $\cosh x$ reaches its minimum 1 for $x = 0$. From $e^{x} \to 0$ for $x \to \infty$ and (13) we see that $\sinh x \to \infty$ for $x \to \infty$.

Most formulas for the ordinary trigonometric functions hold without change for the hyperbolic functions. For instance

$$(15)\quad \cosh(x+y) + \cosh(x-y) = (\tfrac{1}{2})(e^{x+y} + e^{-x-y}) + (\tfrac{1}{2})(e^{x-y} + e^{-x+y})$$

$$= (\tfrac{1}{2})(e^x e^y + e^{-x}e^{-y} + e^x e^{-y} + e^{-x}e^y) = (\tfrac{1}{2})(e^x + e^{-x})(e^y + e^{-y})$$

$$= 2 \cosh x \cosh y.$$

But the reader is warned that

$$\cosh^2 x - \sinh^2 x = 1.$$

The formulas of hyperbolic geometry can be obtained from those of spherical geometry by a simple formal procedure: The angles are in both cases determined by the local euclidean geometry and are therefore treated alike. This means that *all functions of angles are left as ordinary trigonometric functions. But the ordinary trigonometric functions of the sides are replaced by the corresponding hyperbolic functions.* For instance the hyperbolic Theorem of Pythagoras is

$$\cosh c/\rho = \cosh a/\rho \cosh b/\rho,$$

whereas (8) becomes

$$\cos A = \sin B \cosh a/\rho.$$

Thus the angles of a triangle determine also in hyperbolic geometry its sides. ρ is again a positive constant typical for the specific hyperbolic geometry under consideration, but it does not have such a simple euclidean interpretation as in spherical geometry.

That *straight lines in hyperbolic geometry intersect at most once* can be seen as follows: Let the lines g and h form the angle α at P. On g lay off from P the distance x to the point X and let d be the perpendicular distance of X from h. The hyperbolic formula which is analogous to (7) shows that $\sin \alpha = \dfrac{\sinh d/\rho}{\sinh x/\rho}$ or $\sinh d/\rho = \sin \alpha \cdot \sinh x/\rho$. It was proved above that $\sinh x/\rho$ increases monotonically from 0 to ∞ as x goes from 0 to ∞. Therefore $\sinh d/\rho$ tends monotonically to ∞ and so does d because of the monotonicity of $\sinh x$. This implies of course that X will never lie on h when $x > 0$. (In the spherical case X lies on h for $x = \rho\pi$, because $\sin \rho\pi/\rho = 0$.)

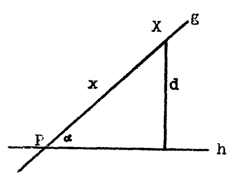

Figure 4

The preceding discussion is nothing but a purely formal procedure for deriving hyperbolic facts from spherical ones. No geometric argument was advanced for justifying this procedure. The next section will indicate how the mobility principle leads to such arguments.

4. *Mobility of figures. A motion of the euclidean plane* Π *or of the sphere* Σ *is a mapping* f *of* Π *(or* Σ*) on itself which preserves distances*, which means that the images $f(P)$ and $f(Q)$ of any two points P,Q of Π (or Σ) have the same distance as P and Q:

$$f(P)f(Q) = PQ \tag{16}$$

As example we consider *rotations*. On Π or Σ choose an arbitrary point R and an oriented angle α. A rotation f about R through α is

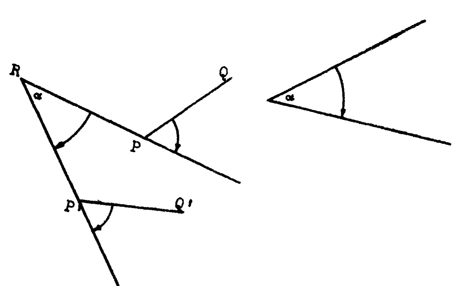

Figure 5

defined as follows: 1) $f(R) = R$ 2) For any point $P \neq R$ we lay off the angle $<PRP' = \alpha$ with RP as initial and RP' as terminal side and choose $P' = f(P)$ such that $RP' = RP$. The reader will easily prove that (16) holds for rotations of Π or Σ.

Reflections in a line give a second example of motions. Let g be a straight line on Π or Σ. For any R on g we put $f(R) = R$. If P is not on g let F be a point of g closest to P (the point F is uniquely determined unless (on Σ) P happens to be a pole to g as equator). On the prolongation of PF beyond F we choose $P' = f(P)$ such that $P'F = PF$. The reader will again easily confirm (16). The present definition of rotations and reflections carry over to hyperbolic geometry.

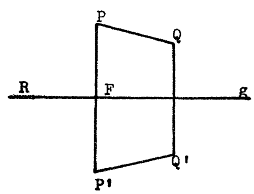

Figure 6

The most general motion can be composed from rotations and reflections. Let ABC and $A'B'C'$ be any two congruent triangles in Π or Σ. If R is any point which has the same distance from A and A', then a rotation f_1 about R through a suitable angle will carry A' into A or $f_1(A') = A$.

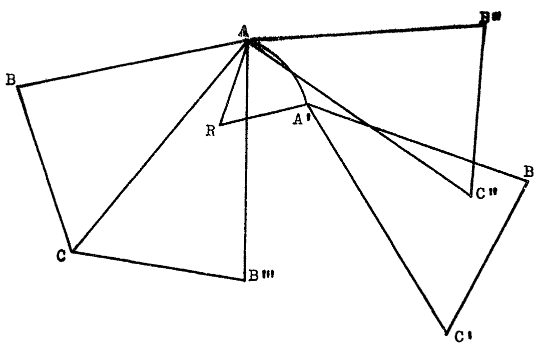

Figure 7

Let $f_1(B') = B''$, $f_1(C') = C''$. Then the triangle $AB''C''$ is congruent to $A'B'C'$ and therefore to ABC. Since $AC = AC''$ a rotation f_2 about A (through $C''AC$) will carry C'' into C or $f_2(C'') = C$. Let $f_2(B'') = B'''$. Since ACB''' is congruent to ACB, the point B''' either coincides with B or (as in the figure) B can be obtained from B''' by a reflection in the line AC. In any case we can move the triangle $A'B'C'$ into the triangle ABC by combining at most two rotations and a reflection.

The spherical case exhibits a phenomenon which is foreign to the other two geometries: a rotation f about R also moves a straight line h, the equator to R as pole, into itself. Or in the standard terminology: f is a *translation* of Σ along h. In the other two geometries translations can be combined from rotations (or reflections) but they are not themselves rotations. They are defined as follows: Let an oriented straight line h and a positive number x be given.

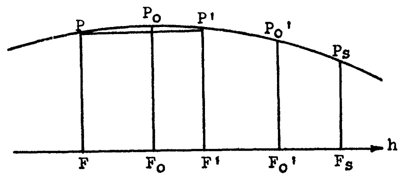

Figure 8

To find the image $P' = f(P)$ of a given point P drop the perpendicular PF from P on h ($P = F$, if P lies on h). From F lay off x in the positive direction on h to the point F'. On h erect at F' a perpendicular $F'P'$ on the same side as P and such that $F'P' = FP$. Then $P' = f(P)$. If s changes $f(P)$ varies on an equidistant curve k to h because every point of k has distance PF from h.

Only in the euclidean case is k a straight line, f is then also a translation along k. In the hyperbolic and spherical cases f cannot be interpreted as translation along any other line but h.

The mobility principle is the requirement that congruent figures can be moved into each other. Here we restrict ourselves to the two-dimensional case. It then suffices to require mobility for triangles. A rigorous form of the principle is this:

(17) *If* $AB = A'B'$, $BC = B'C'$, $CA = C'A'$

then a motion f exists such that $f(A') = A$, $f(B') = B$, $f(C') = C$. Since a combination of motions is again a motion the preceding considerations show that euclidean and spherical geometries satisfy the mobility principle. We want to understand why hyperbolic geometry is the only other geometry in which this principle holds.

The problem presupposes a set of general geometries from which the mobility principle selects the three geometries. How can ''general geometries'' be defined? (17) shows that the concept of distance is essential. We therefore assume that our space is such that for any two points x,y a distance xy is defined and has the following three properties whose content will be clear without further comment:

1) $xy = 0$ means $x = y$

2) If $x \neq y$, then $xy = yx > 0$

3) $xy + yz \geq xz$

These conditions are, of course, still much too weak as a foundation.

For instance, take any set of points and define $xy = yx = 1$ for $x \neq y$ and $xx = 0$. Then properties 1), 2), 3), and (17) are satisfied (to see the latter we define f at the points A', B', C' as in (17) and $f(P) = P$ for $P \neq A'$, B', C', A, B, C, finally $f(A) = A'$; $f(B) = B'$, $f(C) = C'$). It is clear what is wrong with the example: it is impossible to go continuously from one point to another. For our problem *the distance of x and y must be the length of the shortest route from x to y.* In exact language we require

4) If $x = y$ then a curve $z(t)$, $0 \leq t \leq xy$ from x to y (that is $z(0) = x$, $z(xy) = y$) exists such that $z(t_1)z(t_2) = |t_1 - t_2|$

This implies that for $t_1 < t_2 < t_3$

$$z(t_1)z(t_2) + z(t_2)z(t_3) = |t_1 - t_2| + |t_2 - t_3|$$

$$t_2 - t_1 + t_3 - t_2 = t_3 - t_1 = |t_1 - t_3| = z(t_1)z(t_3)$$

so that distances along a shortest route add up (instead of only 3)). Requirement 4) excludes the above example, but it also excludes Σ, if the distances of ordinary space are used instead of the spherical distances along great circles (going through the earth instead of along its surface).

4) guarantees only that y can be reached from s by a curve of length xy. It is also necessary to know that we can go beyond y, at least if y is not too far from x. (On Σ the antipode P' to a given P is a point beyond which it is not possible to go if we start at P). Moreover we need to know that there is only one shortest route from x to y, again only if y is not too far from x. (There are infinitely many different shortest routes from P to P' on Σ). All this can be easily expressed by conditions similar to 4). But 1) to 4) will suffice to give the reader an idea how the problem of defining ''general geometries'' is approached in modern mathematics.

Without going into further details we assume that *every point P is contained in a plane piece in which shortest connections are unique and can be prolonged.* If P is on Σ, a hemisphere with P as pole would be such a piece. It is true that a hemisphere is not a piece of a plane in the usual sense, but it can be bent and stretched into one if we imagine it to be of rubber. After this bending process we define the distance of two atoms x and y as equal to the distance of the same atoms in their original position on Σ. Thus we obtain a plane piece with a spherical geometry. Intrinsically the hemisphere does not differ from the plane piece. A difference appears only if we try to imbed the piece in ordinary space, that is derive its geometry from the geometry of the surrounding space as we did in section 1. In the plane form it cannot be imbedded. The reader will wee that the following argument is entirely intrinsic.

Consider a straight line h and an equidistant curve k to h obtained by translation of a point P along h (see Figure 8). If the feet F;F' of P and P' are such that they have the same distance (and orientation) as the feet F_0, F_0' of P_0 and P_0', then the figures $PFF'P'$ and $P_0F_0F_0'P_0'$ consisting of three straight segments and an arc of k each are congruent, because one figure can be obtained from the other by translations along h. Therefore the arcs PP' and P_0P_0' of k have equal lengths. We see that equal arcs of k correspond to equal distances of their feet on h. Therefore, if F_s is the point on h obtained by laying off the distance s from F_0 in the positive direction and P_s is the corresponding point of k, then the length t of the arc P_0P_s must be proportional to s; the factor of proportionality depends on the distance $b = PF = P_0F_0 = P_sF_s$, or in a formula,

$$t = \phi(b)\cdot s. \tag{18}$$

We have to determine the function $\phi(b)$. For that purpose construct the equidistant curves to h (all on the same side of h) at distances x, $x + y$, and $x - y$ from h where $x > y > 0$. Divide the segment from F_0 to F_s into n equal parts, where n is any even positive integer.

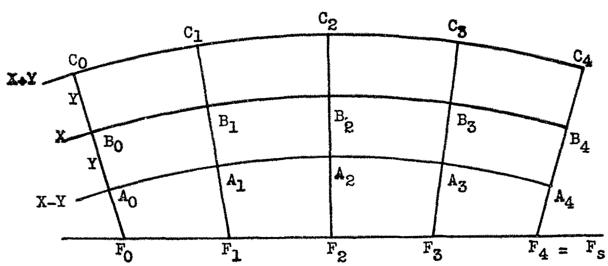

Figure 9

Denote the points of division by F_0, F_1, $F_2, \ldots, F_n = F_s$, and the points corresponding to F_i on the equidistant curves by A_i, B_i, C_i as indicated in the figure which corresponds to $n = 4$. The relation (18) yields for the length of the arc of the equidistant curve

$$\text{length } A_0A_n = \phi(x - y)\cdot s, \quad \text{length } B_0B_n = \phi(x)\cdot s, \tag{19}$$

$$\text{length } C_0C_n = \phi(x + y)\cdot s$$

The quadrangles $A_iC_iC_{i+1}A_{i+1}$ bounded by the arcs A_iA_{i+1} and C_iC_{i+1} of equidistant curves and the straight segments A_iC_i and $A_{i+1}C_{i+1}$ are all congruent. By using the mobility principle we put these quadrangles together in a new way, namely we put every second one upside down, so that A_1 coincides with C_1, etc. as shown in Figure 10.

In the new figure the points $B_0, \ldots, B_n$ lie on a straight line. The length of the middle arc is the same as before, that is $z = \phi(x)\cdot s$ because it consists of the same arcs. The upper and lower arcs consist each of half the previous upper and lower arcs and have therefore

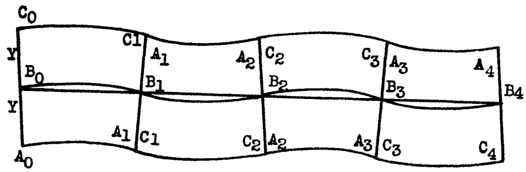

Figure 10

length (compare (19))

$$w = \tfrac{1}{2}(\text{length } A_0A_n + C_0C_n) = \tfrac{1}{2}[\phi(x - y) + \phi(x + y)]\cdot s$$

The segments C_iA_i are still perpendicular to the new middle curve. Therefore as n increases the middle curve tends to a straight segment of length z and the upper and lower curves tend to the equidistant curves to the middle curve at distance y. By using (18) we find

$$w = \phi(y)\cdot z,$$

and substituting the values of w and z and multiplying by 2

$$\phi(x + y) + \phi(x - y) = 2\phi(x)\cdot\phi(y). \tag{20}$$

We have reduced the problem to solving this so-called functional equation for ϕ. Everyone notices at once the trivial solution $\phi(x) \equiv 0$, but this is excluded because $\phi(0) = 1$, which expresses that for $b = 0$ or $x = 0$ the points P_s and F_s coincide so that $t = s$. Another obvious solution is $\phi(x) \equiv 1$. This leads to the euclidean geometry in which P_s moves on a straight line and $t = s$.

From trigonometry the reader is familiar with the identity

$$\cos(x + y) + \cos(x - y) = 2 \cos x \cos y$$

Therefore $\cos x$ and, more generally, $\cos ax$, where a is any constant, is a solution of (20). Formula (15) shows that $\cosh ax$ is also a solution of (20).

It is quite within the reach of a sophomore to see that 0, 1, $\cos ax$, and $\cosh ax$ are the *only continuous solutions* of (20), but the proof will be omitted due to lack of space. If we replace a by $1/\rho$ we can say (compare (2)):

If the mobility principle holds, then the length t of the arc of an equidistant curve to a straight line at distance b has one of the following three forms:

$$t = s, \quad t = \cos b/\rho\cdot s, \quad t = \cosh b/\rho\cdot s$$

The third expression originates from the second by the formal procedure explained in Section 2. But now it was obtained by geometric arguments. In the same way, not only all the hyperbolic formulas can be derived, but the reader sees that the euclidean and spherical formulas appear at the same time and that *no other formulas are compatible with the mobility principle.*

5. *Hyperbolic geometry.* After the procedure of Section 2 has been justified hyperbolic geometry can be legitimately based on it. We describe here, without proofs, some of its properties.

An equidistant curve k to a straight line h is a straight line in euclidean geometry. In spherical geometry it is a circle whose center is on the other side of k from g. Therefore k turns its convex side towards g. In hyperbolic geometry k is neither a circle nor a straight line, but a new kind of curve which turns its concave side towards g as in Figure 8. The segment P_sF_s forms with k (and h) a right angle. Therefore the straight quadrangle $PP'F'F$ has at P and P' angles which are (equal and) smaller than $\pi/2$, so that the angle sum in $PP'F'F$ is less than 2π. This leads to the fact that

In hyperbolic geometry the sum of the angles of a triangle is less than π.

The area of a hyperbolic triangle with angles α, β, γ is proportional to $\pi - \alpha - \beta - \gamma$, and remains therefore below a fixed number although triangles with arbitrary large sides exist (for instance, equilateral ones).

Consider a straight line g in the hyperbolic plane and a point P not on g. Drop the perpendicular PF from P on g and let X traverse one of the rays of g determined by F. The line PX turns monotonically

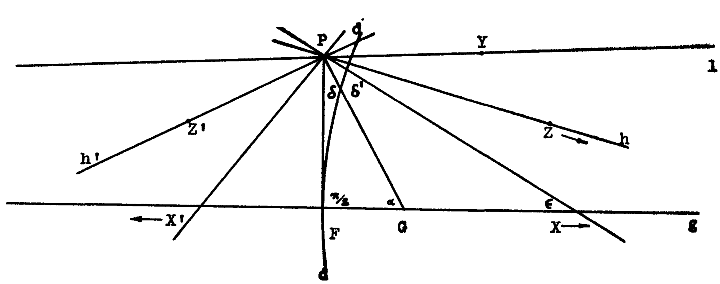

Figure 11

and tends to a limit position h. If G is a fixed point on the same ray as X we find, with the notation of figure 11,

$$\pi/2-(\delta+\delta')>\pi/2-(\delta+\delta'+\epsilon)=\pi-(\alpha+\delta+\pi/2)+\pi-(\delta'+\epsilon+\pi-\alpha).$$

The parentheses on the right are the angle sums in the triangles PFG and PGX respectively. Since these are smaller than π

$$\pi/2 - (\delta + \delta') > \pi - (\alpha + \delta + \pi/2) > 0.$$

Because P, F, G do not depend on X we have

$$\lim \angle FPX = \lim(\delta + \delta') = \angle FPZ < \pi/2$$

By reflecting this figure in the line PF we see: if X' traverses the other ray of g then

$$\angle FPZ' = \lim \angle PFX' = \lim \angle PFX < \pi/2$$

Therefore $\angle ZPZ' < \pi$, so that the limit position h' of PX' is different from h. The lines through P inside $\angle ZPZ'$ intersect g, all the other lines through P do not intersect g. *The parallel axiom does not hold.*

A line l inside the supplementary angle to $\angle ZPZ'$ is usually called a hyper parallel to g. The distance of a variable point y on 1 from g reaches a minimum at some point and tends to ∞ as y tends in either direction to ∞.

If Z traverses h in the indicated direction its distance from g tends to 0. The lines h and h' are called parallels to g. The angle $\beta(d) = \angle FPZ = \angle FPZ'$ which depends only on the distance $b = PF$, is called the *parallel angle* belonging to b. Its value is given by the relation

$$\tan \tfrac{1}{2}\beta(d) = e^{-d/\rho}$$

In spherical geometry a straight line is at the same time a circle (about the pole to the line as equator). In euclidean geometry a straight line is not a circle but it is the limit of circles whose radii tend to ∞. *In hyperbolic geometry a circle with center X through F (Figure 11) does not tend to a straight line as X tends on g to* ∞, but to an important new kind of curve d which is orthogonal *to* all parallels to the orientation of g in which X recedes.

These facts explain why the discovery of hyperbolic geometry had such a deep influence on mathematics. Because of its many new phenomena it freed us from the shackles of euclidean tradition. *It is somewhat richer than the other two geometries.* The following fact elucidates this contention: In three dimensional hyperbolic space the geometry on a sphere is spherical (but its hyperbolic radius is not equal to the ρ belonging to the same sphere if imbedded in euclidean space). As the hyperbolic sphere becomes large it does not tend to a plane, but to a curved surface, whose geometry is euclidean. Thus both the other geometries can be derived from hyperbolic geometry as the spherical from euclidean. On the other hand, three dimensional spherical geometry does not contain models of either of the other two geometries, a result which is made plausible by the finite extent of spherical geometry.

FOOTNOTES

1) The ordinary geometry, with which the reader was first acquainted in high school, is the main subject of Euclid's Elements. This book remained the standard textbook on geometry until the end of the last century. Therefore ordinary geometry is also called euclidean geometry.

2) The parallel axiom states: if a line g and a point P not on g are given, then the plane determined by P and g contains exactly one line through P which does not intersect g. The formulation of the axiom in Euclid's Elements is slightly different, but equivalent.

3) There are other unifying principles; for instance, projective geometry furnishes such a principle.

University of Southern California

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Comment on the review of *Modern Operational Calculus (N. W. McLachlan)* by C. C. Torrance in the Mathematics Magazine.

The review of my book on p. 158, Mathematics Magazine, Vol. XXII, No. 3, Jan.-Feb., 1949 is liable to be misleading. As stated in the preface, the book was written for graduate engineers and physicists. Being an engineer, I should not attempt to write for mathematicians. In my earlier book on Complex Variable and Operational Calculus, I was criticised for lack of rigor, but now the criticism implies that there is too much rigor and too little latitude. I have been of the opinion for some years that Technical Mathematics ought to be sufficiently rigorous for practical purposes. In industry where costly apparatus is designed, and mistakes cannot be tolerated, the applied mathematician must not only be able to get an answer, but also prove that it is correct. This can be achieved only with the aid of adequate rigor. My book is a first attempt to set a standard in this direction. The rigor will have to come sooner or later, so why not now before it is too late!

In writing for engineers it is sometimes necessary to introduce definitions which are unknown to them, although of everyday occurrence to mathematicians. We are told that ''the author's style is very mechanical. Motivation and heuristic are almost completely lacking, and the student is treated as little more than a machine''. This reminds me of the passage in Gilbert and Sullivan's light opera *Mikado*; ''My object all sublime, I will achieve in time, to make the punishment fit the crime, the punishment fit the crime''. I seem to have the distinction of achieving this in *no* time! I am convinced, after many years experience as an engineer, and in teaching engineers, that this is the proper course in dealing with operational calculus. To them it is merely a tool, and as such it should be used with machine like precision. Mathematicians often speak of the beauty of their analysis. To me analysis is merely logical, interesting, mentally stimulating, and useful in practice. I have to look elsewhere for something to which the word 'beauty' can be applied in its usual dictionary sense, e.g. a flower, a picture.

The English language may be interpreted in more than one way. A 'verbal act' which is satisfactory in Britain does not always find favor in America. I have heard Americans criticising eminent British mathematicians for not saying what they mean. We do not find their language ambiguous here, so it appears that English phraseology can have different meanings in the two countries.

Finally, absence of proofs of the theorems in the appendices is due entirely to the exigencies of our present economic position. In Britain it is absolutely essential to keep the size of a book as small as is compatible with intelligibility. Owing to shortage of materials, limited sales, and the cost of production, we are unable to produce advanced mathematical texts of 400 pages or more.

N. W. McLachlan

Introduction to the Theory of Probability and Statistics by Neils Arley and K. Rander Buch. New York (John Wiley and Sons) $4.00; London (Chapman and Hall). 1950.

The book under review is a second volume in the Applied Mathematics Series of John Wiley and Sons, and is essentially a translation (by one of the authors) of an earlier Danish edition.

The authors have tried to avoid the pitfalls of differences in languages by giving in footnotes the idiomatic terminology in both French and German for many important concepts. They may be forgiven for referring to ''partial integration'' when ''integration by parts'' is apparently meant.

The reviewer is reminded of an occasion when he translated the German ''Hoch Schule'' into the literal English equivalent ''high school'', only to learn that ''junior college'' was a much better translation. It may be that the authors of the book under review have translated too literally a Danish title into the English ''Introduction'' when a much more impressive word is called for. In any event, the word ''Introduction'' seems to have been poorly selected. Only the most hearty students in this country with no previous acquaintance with probability and statistics would find this book suited to their needs and talents. The subject matter of the book is elementary in the sense that the foundations of the theory are not based on set-theoretic concepts as the authors note might have been the case, and also in the sense that separate presentations are made of the discrete and continuous cases instead of combining these by means of the concept of the Stieltjes integral, a possibility which the authors also mention.

The book would seem to be best adapted to senior and graduate students in mathematics and the exact sciences. Vector notation is used in many of these examples, and matrix notation is also used (with an appendix explaining the terminology). Most of the sections using these and similar concepts are starred and may be omitted without destroying the continuity of thought in the unstarred sections. Many of the equations are followed by an implied directive: ''(Check this)'', as well as many of the stated numerical results. Some familiarity with with mathematical concepts would thus be necessary for effective use of the book.

The text is scattered with exercises and examples, some of which are given in outline form only, and there is a list of 90 problems at the end of the book. Many discrete and continuous theoretical distributions are dealt with. The careful student who works through the material of this book will indeed be ready for advanced work in statistics.

A typographical feature of the book deserves no praise from the point of view of this reviewer. In some cases, an expression of the form $F(w, x, y, z, t)$ starts on one line and ends on another line. To the reviewer, this appears as improper mathematical hyphenation. In most cases, the resulting form was a term in an equation and the equation could have been displayed with only slight loss in space and with considerable improvement in typography.

Perhaps the most refreshing feature of the book is the very excellent treatments on the nature of mathematical models and the possibilities of comparing these models with perceptual physical models of the external world.

University of Texas Robert E. Greenwood

Set Functions. By Hans Hahn and Arthur Rosenthal. The University of New Mexico Press, 1948. ix + 324 pages.

This volume is based upon manuscripts left by the late Professor Hahn, revised, edited and brought up to date by Professor Rosenthal. Its point of view is extremely general, since the domain of the set functions considered is usually any arbitrary set of elements. At times the sets must be specialized in order to obtain the desired theorems. For example, in discussing differentiation, a metric space is assumed and in certain cases a euclidean n-space is needed.

The object of the authors is to generalize as much as possible the theorems of analysis. Some generalizations seem to be more apparent than real, but the treatment of integration and differentiation includes extensive generalizations of theorems which are well established for euclidean spaces.

After an introduction in which sets and topological and metric spaces are defined and illustrated, there follow five chapters on Set Functions, Measure, Measurable Functions, Integration and Differentiation. In addition to a brief bibliography at the end of the book, more comprehensive references are given at the end of each section.

The book is recommended for reference purposes or for use as a text in a graduate course. An excellent feature is the frequency of examples to show that certain restrictions in the hypotheses are essential. This should be extremely helpful to readers who are unfamiliar with the general concepts involved.

University of Buffalo Harry M. Gehman

STANDARDS - FOR WHAT? by James E. Foster (See ''High Standards - Sacred and Profane'' Mathematics Magazine Vol. XXIII No. 4, Mar.-April.

Unlike sin, which every one is against, we are all for standards. While an occasional appeaser may suggest that standards be reasonably related to the ability of people to conform, to most of us this smacks of compromise with principle. Standards we are for, and the higher the better.

This is neither the time nor the occasion to make a Quixotic attack on the windmill of high standards. Nevertheless, not even the passionate devotee to standards denies it should be subject to critical scrutiny, at least in its specific aspects. If the standard is sound, criticism will vindicate it and in the process add to its prestige. If it is vulnerable, its weakness needs exposing that it may be corrected.

Whatever may be said in criticism of modern education, no one can seriously deny that our schools are turning out graduates who are competent within the fields of their study. The neophyte doctor, lawyer, engineer, accountant, stenographer, radio mechanic, laboratory technician, ballroom dancer, fresh with diploma or certificate of study, can function competently in the field for which he or she was trained. The arts or science student with less practical aspirations acquires a sound basic knowledge of his major field, plus a broad background of other information.

For that we must credit existing educational standards. In post graduate professional colleges, in trade schools that advertize glittering promises to attract students, from kindergarten up, there are standards levelled against incompetency. No one can seriously quarrel with such an objective or deny the general effectiveness of the result.

One may wonder, however, whether in our quest for competence we may have handicapped what should be a major objective of education in the best sense of the word.

More than ever before, people become competent through formal education. Apprenticeship is today more of a vestige of the past than a means of learning a trade. Where colleges and universities once limited their curricula to the classics, at most a half dozen foreign languages, literature, and a few basic sciences, the present day college student can study in virtually any field that interests him. Whatever one's aspiration, the first step in realizing it is to go to school and learn how.

Though we probably over-value such training, the importance we place on knowing the answers is of itself to the good. We live in a complicated world in which thousands of different types of technicians are needed. We appreciate them every time, for example, the elevator operators in New York or the bridge tenders in Chicago threaten to strike. If we are to continue to live as we are, we must have a continuing supply of all sorts of technicians, who have acquired their know-how somewhere along the education-experience front.

All of which is to say there is nothing wrong with schools training the people we need trained - from gadget operators to atomic physicists. And, while we are at it, it is only prudent to have high standards of competence. But, we also need a sense of proportion in all this. In our recognition of the value of knowledge and in our utilization of the school as the means of acquiring it, we seem to be confusing scholastic with intellectual virtues. To credit a man with ''common sense'' is virtually to insult him by inferring that, while he may be bright in his way, he is nonetheless an ignoramus and isn't it a shame he didn't get an education. You do not compliment him by saying he has ''sound judgment''. To do that, you must refer to him as ''well trained'' and ''well informed''.

Again, well trained people who are up on their facts are vital to modern civilization. But, if modern civilization needs no more than that in the way of brain power, the people for whom that civilization exists do. Skills and knowledge, no matter how extensive they may be, can do nothing more than maintain a status quo. But, whether we like it or not, life is dynamic with change being one of the few persistent phenomena of history. Ability to maintain an existing order may be desirable as long as that order exists. But, unless that ability can adapt itself to change, its value is transitory.

Mere adaptation, however, is not enough. The direction change takes must be intelligently determined, if civilization is to progress. That is what happened in the past. The history of man's control over nature can be written in terms of the deliberate manipulation of natural forces to achieve predetermined ends.

When this happens we have progress, which is nothing more than the result of purposeful imagination. No one would argue against its desirability. One might even think it worth encouraging through our educational programs. Some one might reply that we do, and for evidence point to the original contribution to knowledge supposedly a part of every doctorial thesis.

That is to say, if only by implication, that no serious effort is made to stimulate imaginative thinking until the student is working for a doctorate. Even then, the pay-off is usually for painstaking drudgery. The typical doctorial thesis - ''an original contribution to knowledge'' - is a monumental compilation of quotations and of tabular material to which an assortment of statistical gymnastics have been applied. It is duly accepted by a committee that recognizes diligent and sustained effort when it sees it; a copy is deposited in the university library, there to collect dust (or to be kept free from it by periodic dusting) not because it is an undiscovered jewel, but because it is an exercise in research and nothing more.

Possibly this situation is inevitable. It may be that for every 100 persons who are capable of being ''well trained'' and becoming ''well informed'', there is but one who can produce something original. Certainly, under such circumstances we must train the 100, but in the process we need not stifle the one into conformity.

Does some one object - contend that, since imagination must work on a residue of knowledge, the greater that residue, the greater the opportunity for effective imagination; that the well trained, the well informed is more likely to do something with his imagination than is the ignoramus or even the superficially informed?

Agreed. Nevertheless, important work has been done in a number of fields by people of decidedly limited knowledge. The Pasteur who revolutionized the practice of medicine was not a physician or even a biologist, but a chemist. And, George Boole, whose *Laws of Thought* must be rated as one of the two or three major contributions of mathematics to nineteenth century scholarship, had less formal training in that field than a present-day high school graduate.

A Pasteur and a Boole do not, however, invalidate the contention that the better informed a person is, the better equipped he is for creative work. But, they do point out that one can contribute significantly to a specialized field before one enters the last laps of the educational process, and by implication suggest that originality needs nurtuting all along the line. In this connection, consider the number of men who have made important contributions to mathematics before they reached the age of thirty, some before they were even twenty-five.

But, with standards that emphasize ''training'' and that measure education quantitatively, and with an educational process that has no place for original work until a student starts work for a doctorate, creative thinking below the advanced stages of graduate work is at best a profitless avocation; and it may be a handicap. By and large, the undergraduate and the candidate for a master's degree progress through absorbing what they are exposed to; the doctorial candidate knows he can get by through a continuation of this process together with making a study that is patently laborious without necessarily being contributory to knowledge in any real sense. Success, as determined by modern education, depends on academic degrees; one gets such degrees by systematically giving back in answers to examination questions a part of what is handed one through textbooks, lectures, and outside readings. As a matter of prudence, it is all very well to toy with original thought, so long as it doesn't interfere with learning the answers.

Is this situation limiting the output of original thought? There seems to be agreement that the output is relatively low. In this connection, George Sarton said: ''It is quite certain that the number of original mathematicians has not increased in proportion to the number of well trained mathematicians, or to the availabilities for mathematical research''. It is true his explanation of this situation is different from that implied in this paper. According to him ''mathematical theory is not very much determined by external circumstance. The main factor is the availability of creative genius, which cannot be controlled''.

''Creative genius'' undoubtedly is needed. To thrive, however, creative genius must have the encouragement of stimulating atmosphere. It is hardly coincidence that few, if any, creative geniuses in mathematics flourished without company. There was the golden era of Grecian culture. Descartes and Fermat were contemporaries, as were Newton and Leibnitz. The early and middle nineteenth century had a Gauss, a Bolyai, a Lobachevsky, and a Boole. Hardly happenstance.

Isn't it more reasonable to blame the relative paucity of original mathematicians on educational standards that set as their sole goal a high level of competence, that through their disregard of originality, actually handicap it? No one will argue against competency; or against originality, either. But, our educational standards uncompromisingly protect the one and at most give lip service to the other. When we realize that, in the long pull, originality is probably the more important of the two, we will do something to encourage it throughout the entire educational process, without any sacrifice of competence.

James E. Foster

PROBLEMS AND QUESTIONS

Edited by

C. W. Trigg, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject-matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by such information as will assist the editor. Ordinarily, problems in well-known text-books should not be submitted.

In order to facilitate their consideration, solutions should be submitted on separate, signed sheets within three months after publication of the problems. Readers are invited to offer heuristic discussions in addition to formal solutions.

All manuscripts should be typewritten on 8½" by 11" paper, double-spaced and with margins at least one inch wide. Figures should be drawn in india ink and in exact size for reproduction.

Send all communications for this department to *C. W. Trigg, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, Calif.*

PROPOSALS

70. *Proposed by J. S. Cromelin, Clearing Industrial District, Chicago, Illinois.*

Smith, Brown, Jones, and Robinson were married to Minnie, Maggie, Maud, and Mabel, but not in that order. Each one bought as many pineapples as he or she paid cents apiece for them. Mabel bought four times as many as Mr. Jones, and Mr. Brown paid three cents apiece more than his sister, Maud. Each wife spent $1.05 less than her husband, and Mrs. Robinson spent more than Mrs. Smith. From these data the full names of the two ladies visiting Hawaii at the time may be determined. Who were they?

71. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

Using each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 once and only once, form three numbers, m, n, r, such that the sixth power of r equals the sum of m and n.

72. *Proposed by D. L. MacKay, Manchester Depot, Vermont.*

Construct a triangle given, in position, a vertex A, the foot D of the corresponding altitude, and a point M which divides a second altitude in a given ratio $p:q$.

73. *Proposed by W. B. Carver, Cornell University.*

If P_1P_2 is any arc of a parabola, show that the radius of curvature at some interior point of the arc is greater than the length of the chord P_1P_2.

74. *Proposed by Samuel Skolnik, Los Angeles City College.*
Prove that the sum of any finite number of dissimilar pure quadratic surds is irrational.

75. *Proposed by W. E. Byrne, Virginia Military Institute.*
Consider the real roots in the interval $-\pi \leqq \phi < \pi$ of the equation

$$f(\phi) = \omega^2 b \sin 2\phi - 3(g \sin \phi - \omega^2 a \cos \phi) = 0,$$

where ω, a, b, g are positive constants. Show that there may be two or four distinct roots, or three distinct roots of which one is double. What relations must hold among the constants to give a double root?

76. *Proposed by Howard Eves, Oregon State College.*
Let A, B, C be three fixed points in the Gauss plane. What is the locus of a variable point Z if (A, B, C, Z) is pure imaginary?

SOLUTIONS

Resultant of Two Coplanar Forces

25. [Nov. 1948] *Proposed by K. E. Cappel, San Francisco, California.*
Three points, A, B and C, are located on a straight line. A force f_1, of known magnitude and direction, passes through B. A second force f_2, coplanar with f_1 and ABC, and of known magnitude but unknown direction, is to pass through the point A in such a way that the resultant of f_1 and f_2 passes through the point C. Find the direction of f_2.

Solution by Howard Eves, Oregon State College. Let θ be the non-obtuse angle between AB and the line of action of f_1. Let P be the intersection of the lines of action of f_1 and f_2 and denote the magnitudes of these forces by b and a respectively. Also, let α, β represent angles CPB, APC, and ϕ the non-obtuse angle between AB and AP. Then, by the law of sines,

$$PC/CB = \sin\theta/\sin\alpha, \quad PC/AC = \sin\phi/\sin\beta, \quad a/b = \sin\alpha/\sin\beta,$$

whence

$$\sin\phi = (CB/AC)(b/a)\sin\theta.$$

If $(CB/AC)(b/a)\sin\theta \leqq 1$, angle $\phi \leqq \pi/2$ can easily be constructed – with euclidean tools, in fact. Then, if $\phi \neq \theta$ and $\phi \neq \pi/2$, there exist two suitable points P on the line of action of f_1. We may take f_2 as acting along either one of the lines AP.

Thus there are, in general, two solutions to the problem. However, if $\phi = \pi/2$ or if $\phi = \theta \neq \pi/2$, there is only one solution, and if $\phi = \theta = \pi/2$ or $(CB/AC)(b/a)\sin\theta > 1$, there is no solution.

Also solved by P. N. Nagara, College of Agriculture, Thailand; W. I. Thompson, Los Angeles City College; and the proposer.

An Integral Involving Hyperbolic Functions

46. [Nov. 1949] *Proposed by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.*

Integrate

$$I(x) = \int \frac{a \sinh^3 x + b \cosh^3 x}{c \sinh x + f \cosh x}\, dx$$

where a, b, c, f are constants.

I. *Solution by the Proposer.* Let $c = r \sinh u$, $f = r \cosh u$, where $r^2 = f^2 - c^2$. Then $c \sinh x + f \cosh x = r(\cosh u \cosh x + \sinh u \sinh x) = r \cosh(u + x)$. Let $v = u + x$, $dv = dx$ and the given integral may be written

$$\int \frac{a(\frac{f}{r} \sinh v - \frac{c}{r} \cosh v)^3 + b(\frac{f}{r} \cosh v - \frac{c}{r} \sinh v)^3}{r \cosh v}\, dv. \quad (1)$$

After expanding the numerator of (1) and dividing by the denominator, we restore the variable x and by means of hyperbolic identities write (1) as

$$I(x) = \int \frac{bc^3 - af^3}{(f^2 - c^2)^2} \frac{c \cosh x + f \sinh x}{c \sinh x + f \cosh x}\, dx + \int \frac{bf - ac}{2(f^2 - c^2)} \cosh 2x\, dx$$

$$+ \int \frac{af - bc}{2(f^2 - c^2)} \sinh 2x\, dx + \int \frac{bf^3 - ac^3 + 3fc(af - bc)}{2(f^2 - c^2)^2}\, dx,$$

the integral being then

$$I(x) = \frac{bc^3 - af^3}{(f^2 - c^2)^2} \ln(c \sinh x + f \cosh x) + \frac{bf^3 - ac^3 + 3fc(af - bc)}{2(f^2 - c^2)^2}\, x$$

$$+ \frac{bf - ac}{4(f^2 - c^2)} \sinh 2x + \frac{af - bc}{4(f^2 - c^2)} \cosh 2x + C.$$

II. *Solution by M. Morduchow, Polytechnic Institute of Brooklyn, N.Y.*

Introduce the substitution of variables

$$y = e^{2x} \quad (1)$$

Then, since by definition

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

the integral I becomes:

$$I(x) = \frac{1}{8}\int \frac{a(y-1)^3 + b(y+1)^3}{y^2[c(y-1) + f(y+1)]}\,dy \qquad (2)$$

Case A: If $f - c = 0$, then the denominator of the integrand in (2) becomes $(c + f)y^3$, and the integration in (2) can be carried out by expanding the numerator algebraically, and integrating term by term. The final result thus obtained is [K is an arbitrary constant]:

$$I(x) = \frac{1}{8}\frac{1}{(f+c)}\left[(a+b)e^{2x} + 6(b-a)x - 3(a+b)e^{-2x} - \frac{1}{2}(b-a)e^{-4x}\right] + K.$$

Case B: If $f + c = 0$, then the denominator in (2) becomes $(f - c)y^2$, and the integration can be performed as in Case A. The final result is:

$$I(x) = \frac{1}{8}\frac{1}{(f-c)}\left[\frac{(a+b)}{2}e^{4x} + 3(b-a)e^{2x} + 6(a+b)x - (b-a)e^{-2x}\right] + K.$$

Case C: If $f - c \neq 0$, $f + c \neq 0$, then by the use of partial fractions, the integral (2) can be written in the form:

$$I = \frac{1}{8}\frac{1}{(f+c)}\int \left[(a+b) + \frac{A+By}{y^2} + \frac{C}{y+j}\right] dy \qquad (3)$$

where

$$j = \frac{f-c}{f+c},\quad A = \frac{b-a}{j},\quad B = \frac{3(a+b)j - (b-a)}{j^2}$$

$$C = \frac{(b-c)(3j^2+1) - (a+b)j(3+j^2)}{j^2}.$$

The integration in (3) can be readily carried out term by term, and the final result, after substitution for y by means of (1), is:

$$I(x) = \frac{1}{8}\frac{1}{(f+c)}\left[(a+b)e^{2x} - \frac{(b-a)}{j}e^{-2x} + 2\frac{3(a+b)j - (b-a)}{j^2}x\right.$$
$$\left. + \frac{(b-a)(3j^2+1) - (a+b)j(3+j^2)}{j^2}\log(e^{2x}+j)\right] + K.$$

Morduchow's result may be worked into the form given by Thomas, the relation between the arbitrary constants being

$$C - K = (bc^3 - af^3)\ \ln[2/(f+c)]/(f^2 - c^2)^2.$$

Also solved by W. R. Talbot, Jefferson City, Mo.; W. I. Thompson, Los Angeles City College; and the proposer [in a second way, using the substitution, $\sinh x = 2t/(1-t^2)$].

Four Parallel Cevians of a Tetrahedron

47. [Nov. 1949] *Proposed by N. A. Court, University of Oklahoma.*

Four parallel cevians AA', BB', CC', DD' of a tetrahedron (T) = $ABCD$ are divided by the points P, Q, R, S, internally in the same ratio k. For what value of k will the four points P, Q, R, S, be coplanar?

I. *Solution by Howard Eves, Oregon State College.* We shall employ the easy lemma: The line segment through the intersection of the diagonals of a trapezoid, drawn parallel to the bases of the trapezoid, and cut off by the sides of the trapezoid, is bisected by the intersection of the diagonals.

Now one of the points A', B', C', D' must lie inside the corresponding face of $ABCD$; let A' be that point. Let AA' cut plane $B'C'D'$ in A''; let BA', $B'A''$ cut DC and $D'C'$ respectively in E, E'; and let $D'C$ and $C'D$ intersect in L. Now planes $D'BC$ and $C'BD$ intersect in LB and also each contains point A. Therefore L, A, B are collinear and E, A, B' are collinear. It also follows that L, E, E' are collinear; in fact, by the lemma, L is the midpoint of EE'. Since LB and EB' meet $A'A''$ at A it is easily shown that LB' and $E'B$ meet $A'A''$ in a point H. By the lemma H is the midpoint of AA'', and A is the midpoint of HA'. Therefore $AA' = (A''A')/3$. Now we have

$$A'P/PA = B'Q/QB = C'R/RC = D'S/SD = k.$$

Since plane QRS cuts $A''A'$ in a point P' such that $A''P'/P'A' = k$, it follows that P' will fall on P if and only if $k = 3$.

For the analogous property in the plane, $k = 2$. See, *Mathesis*, page 210, (1901).

II. *Solution by L. M. Kelly, Michigan State College.* Let the barycentric coordinates of A' be $(0, x_2, x_3, x_4)$ with $x_2 + x_3 + x_4 = 1$. Then the coordinates of any point on the line AA' will be (a, x_2, x_3, x_4) where a is a parameter. In particular, the point at infinity on this line will have $a = -(x_2 + x_3 + x_4) = -1$. Thus the barycentric coordinates of the point at infinity on the line AA' are $(-1, x_2, x_3, x_4)$ and the coordinates of any point on the line joining this point to B are $(-1, x_2 + b, x_3, x_4)$ where b is a parameter. In particular, the coordinates of B' are $(-1, 0, x_3, x_4)$. Similarly those of C' and D' are $(-1, x_2, 0, x_4)$ and $(-1, x_2, x_3, 0)$. The point A'' on the line AA' for which $AA''/A''A' = k$ is easily seen to have coordinates (k, x_2, x_3, x_4). Similarly, the coordinates of B'', C'', D'' are $(-1, kx_2, x_3, x_4)$, $(-1, x_2, -kx_3, x_4)$ and $(-1, x_2, x_3, -kx_4)$. The necessary and sufficient condition that these four points be on a plane is that

$$\begin{vmatrix} k & x_2 & x_3 & x_4 \\ -1 & -kx_2 & x_3 & x_4 \\ -1 & x_2 & -kx_3 & x_4 \\ -1 & x_2 & x_3 & -kx_4 \end{vmatrix} = 0.$$

It follows that $k = 3$ or -1. Since -1 leads to points at infinity, the desired k is 3.

Also solved by P. D. Thomas, Washington, D. C.

Centroid of an Arc of the Clothoid

48. [Nov. 1949] *Proposed by Howard Eves, Oregon State College.*

The centroid of any arc of a transitional spiral coincides with the external center of similitude of the osculating circles of the extremities of the arc. (A transition spiral is a curve whose curvature varies directly with the arc length.)

Solution by the Proposer. Let the spiral have its initial point at the origin and its initial tangent along the positive x-axis. Let r be the radius of curvature, s the arc length measured from the origin, and ϕ the inclination of the tangent line for a general point on the curve. Then, where c is a constant of proportionality,

$$c = sr = s(ds/d\phi).$$

Separating variables and solving we find $s^2 = 2c\phi$, or

$$s = 2K\phi^{\frac{1}{2}}, \quad K = \sqrt{2c}/2.$$

Hence

$$dx = ds \cos\phi = K\phi^{-\frac{1}{2}} \cos\phi \, d\phi,$$

$$dy = ds \sin\phi = K\phi^{-\frac{1}{2}} \sin\phi \, d\phi.$$

The following useful relations are easily found:

$$dy/dx = \tan\phi, \quad d^2y/dx^2 = (\phi^{\frac{1}{2}} \sec^3\phi)/K,$$

$$r = [1 + (y')^2]^{3/2}/y'' = K\phi^{-\frac{1}{2}},$$

and, if (α, β) is the corresponding center of curvature,

$$\alpha = x - r\sin\phi = x - K\phi^{-\frac{1}{2}}\sin\phi,$$

$$\beta = y + r\cos\phi = y + K\phi^{-\frac{1}{2}}\cos\phi.$$

Now let the extremities of an arc s_{12} be the points (x_1, y_1) and (x_2, y_2), with corresponding radii and centers of curvature r_1 and r_2, (α_1, β_1) and (α_2, β_2). Designate the concerned external center of similitude by (h, k), and the concerned centroid by $(\bar{h}, \bar{k})$. Then

$$h = \frac{-r_2\alpha_1 + r_1\alpha_2}{-r_2 + r_1} = \frac{x_2\phi_2^{\frac{1}{2}} - x_1\phi_1^{\frac{1}{2}} - K(\sin\phi_2 - \sin\phi_1)}{\phi_2^{\frac{1}{2}} - \phi_1^{\frac{1}{2}}}.$$

Similarly

$$k = \frac{y_2\phi_2^{\frac{1}{2}} - y_1\phi_1^{\frac{1}{2}} + K(\cos\phi_2 - \cos\phi_1)}{\phi_2^{\frac{1}{2}} - \phi_1^{\frac{1}{2}}}.$$

But

$$ds = K\phi^{-\frac{1}{2}}d\phi, \quad s_{12} = 2K(\phi_2^{\frac{1}{2}} - \phi_1^{\frac{1}{2}}),$$

whence

$$\overline{h} = \left[\int_{\phi_1}^{\phi_2} x\ ds\right]/s_{12} = \frac{\int_{\phi_1}^{\phi_2}\left[K\int_0^{\phi}\phi^{-\frac{1}{2}}\cos\phi\ d\phi\right]K\phi^{-\frac{1}{2}}d\phi}{2K(\phi_2^{\frac{1}{2}} - \phi_1^{\frac{1}{2}})}.$$

Integrating the numerator by parts, by setting

$$u = K\int_0^{\phi}\phi^{-\frac{1}{2}}\cos\phi\ d\phi \text{ and } dv = \phi^{-\frac{1}{2}}d\phi,$$

we readily find that $\overline{h} = h$. Similarly $\overline{k} = k$, and the theorem is established.

Alan Wayne reports that in E. Pascal, *Repertorium der Hoeheren Mathematik*, Vol. II (Part I), page 478, this theorem for the ''Klothoide'' is stated: ''Der Schwerpunkt eines beliebigen Bogens ist äusserer Ähnlichkeitspunkt der beiden Krümmungskreise in der Endpunkten des Bogens.'' The proof is referred to Cesàro, *Nouv. Ann. Math.* (3), 5, 511, (1886). For further properties of the transition spiral (or clothoid) see J. O. Eichler and Howard Eves, *Transactions of the American Society of Civil Engineers*, III, 959 - 1010, (1946), where an extensive bibliography will be found. Also, for the location of the pole of the spiral, see *American Mathematical Monthly*, 52, 278, (1945).

Adjoint Circles of a Triangle

49. [Jan. 1950] *Proposed by F. C. Gentry, Arizona State College at Tempe.*

The product of the radii of either group of adjoint circles of a triangle is equal to the cube of the circumradius.

Solution by D. L. MacKay, Manchester Depot, Vt. To obtain the direct group of adjoint circles of triangle ABC one constructs on sides a, b, c segments of circles capable of containing supplements of angles C, A, B, respectively. These are tangent respectively to b, c, a.

Let r_1, r_2, r_3 be the radii of adjoint circles having a, b, c as chords. Then $r_1 = a/2\sin C$, $r_2 = b/2\sin A$, and $r_3 = c/2\sin B$. Hence $r_1r_2r_3 = abc/8\sin A\sin B\sin C = R^3$, since $R = a/2\sin A = b/2\sin B = c/2\sin C$. A similar argument holds for the indirect group of adjoint circles.

Also solved by Howard Eves, Oregon State College; W. R. Talbot, Jefferson City, Missouri; P. D. Thomas, Washington, D. C.; and the proposer.

A Doubly True Addition

50. [Jan. 1950] *Proposed by Alan Wayne, Flushing, N.Y.*

In the addition *THREE* + *EIGHT* + *NINE* = *TWENTY* no two different letters represent the same digit. Find the value of each letter and show that with $R > G$ the solution is unique.

Solution by R. M. Swesnik, Anderson-Prichard Oil Corporation, Oklahoma City. Since the sum of two unequal digits plus 2 is less than 20, $T = 1$. Then $1 + E + (1 \text{ or } 2) = 1W$. Hence $E = 7$, 8 or 9. But $E \neq 7$, since $H + I + N + 2 \leqq 9 + 8 + 6 + 2 = 25 < 27$. Nor is $E = 9$, for if it were $E + T + E = 19$, but $Y \neq E$. Therefore $E = 8$, $Y = 7$ and $W = 0$, since $W \neq T$.

Considering the unused digits, $14 < 8 + H + N + 1 < 24$, so $8 + H + N + 1 = 21$, whence $H + N = 12$. The only possible values of H and N are $H = 3$ or 9 and $N = 9$ or 3.

Now R, G and I must be chosen from 2, 4, 5 and 6, so $13 \leqq R + G + I + 2 < 17$. It follows that $N = 3$ and $H = 9$. Then since $H + I + N + 1 = 18$, $I = 5$. Then $R + G + 5 + 2 = 13$, and since $R > G$, $R = 4$ and $G = 2$.

Accordingly, the unique reconstruction of the addition is

$$19488 + 85291 + 3538 = 108317.$$

Also solved by Monte Dernham, San Francisco, Calif.; F. L. Miksa, Aurora, Ill.; P. N. Nagara, College of Agriculture, Thailand; W. R. Talbot, Jefferson City, Mo.; and the proposer.

The proposer also offered *THREE* + *THREE* + *FIVE* = *ELEVEN*, which has two solutions if none of the letters represents zero. For other digit restorations involving addition see *The American Mathematical Monthly*, **40**, 176, 424, (1933); **41**, 391, (1934); **42**, 47, 323, 445, (1935); **44**, 540, (1937); **46**, 173, (1939); **54**, 413, (1947).

Sum of Reciprocals Equal to Unity

51. [Jan. 1950] *Proposed by E. P. Starke, Rutgers University.*

Find the smallest number (>1) of distinct, odd positive integers such that the sum of their reciprocals is unity. Determine such a set.

Solution by O. H. Hoke, University of North Carolina. Let $S(x_1, x_2, \ldots, x_k) \equiv \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_k} = 1$. For $k = 9$,

$$S(3,\ 5,\ 7,\ 9,\ 11,\ 15,\ 21,\ 165,\ 693) = 1.$$

Two additional solutions are obtained by replacing x_8 and x_9 in this set by 231 and 315, and by 135 and 10395. Obviously, the set S must contain an odd number of elements. No set S exists for $k \leqq 7$ since

$$S(3, 5, 7, 9, 13, 15, 17) < 1,$$

determining the first five elements of S as 3, 5, 7, 9, 11; but clearly,

$$S(3, 5, 7, 9, 11, 13, x_7) \neq 1,$$

and

$$S(3, 5, 7, 9, 11, 15, x_7) \neq 1$$

for every x_7; and finally,

$$S(3, 5, 7, 9, 11, 17, 19) < 1.$$

Also solved by F. L. Miksa, Aurora, Ill.; P. N. Nagara, College of Agriculture, Thailand; and the proposer.

For $k = 9$, Nagara gave two sets in addition to the first two above:

$$S(3, 5, 7, 9, 11, 15, 33, 45, 385) = 1,$$

and

$$S(3, 5, 7, 9, 11, 15, 35, 45, 231) = 1.$$

Miksa also gave for $k = 11$,

$$S(3, 7, 21, 27, 35, 45, 63, 105, 135, 189, 315) = 1,$$

which he obtained by starting with the divisors of the smallest odd abundant number, 945, and choosing a group whose sum is 945.

W. B. Carver suggested that the problem becomes more difficult if one arbitrarily excludes some of the smaller integers – as, for instance, excluding 3 and starting with 5. In that case the minimum value of k is 21, for which one set is $S(5, 7, 9, 11, 13, 15, 17, 21, 33, 35, 39, 45, 51, 55, 63, 65, 77, 85, 221, 58905, 85085) = 1$.

The proposer remarked that Paul Erdös proposed (some time ago) "Can one represent 1/3 as the sum of reciprocals of different squares?" (The simplest set is the squares of 2, 5, 6, 10, 15, 30). Erdös adds, "Every rational number $<\pi^2/6$ is the sum of reciprocals of different squares. My proof is not quite simple."

A Plane Through the Incenter of a Tetrahedron

52. [Jan. 1950] *Proposed by Victor Thebault, Tennie, Sarthe, France.*

If a plane cuts the surface and volume of a tetrahedron into two equivalent parts, this plane passes through the center of the inscribed sphere of the tetrahedron.

Solution by Howard Eves, Oregon State College. Let p be a plane cutting both the surface S and the volume V of tetrahedron $ABCD$ into two equivalent parts. There is no loss in assuming the tetrahedron lettered so that p cuts the edges AB, AC, AD interiorly in B', C', D'. Let M be the intersection of p and the line determined by A and the incenter I of $ABCD$. Let r be the distance of M from each of the faces ABC, ACD, ADB. Then

$$(1/2)V = (1/3)r(AB'C' + AC'D' + AD'B') = (1/6)rS,$$

and it follows that r is equal to the inradius of $ABCD$, and M coincides with I. This proof does not establish the existence of a plane p.

Also solved by Howard Eves in a second way and by D. L. MacKay, Manchester Depot, Vt. MacKay called attention to the two-dimensional problem – to draw a line which divides a triangle into two parts which are equivalent and isoperimetric – set by G. Lemaire in *Vuibert's Journal de Mathèmatic èlèmentaires*, No. 7293. Also see *L'Intermediaire des Mathèmaticiens*, **17**, 268; **18**, 139-143, 234; and *Mathesis*, **42**, 319.

Pythagorean Triangles with Equal Perimeters

53. [Jan. 1950] *Proposed by F. L. Miksa, Aurora, Ill.*

Find the shortest perimeter common to four different primitive Pythagorean triangles.

Solution by the Proposer. It is known that generators s, t with $(s, t) = 1$, $s \not\equiv t \pmod 2$ and $s > t > 0$ yield a primitive Pythagorean triangle with sides $a = 2st$, $b = s^2 - t^2$, $c = s^2 + t^2$. Then the semi-perimeter $P/2 = su$, where $u = s + t$, $s < u < 2s$, $(s, u) = 1$. These conditions imply that $\sqrt{P}/2 < s < \sqrt{P/2} < u < \sqrt{P}$. This gives limits for s and u that are a direct aid in the numerical work of forming four different factorizations of $P/2$ into two parts. $P/2$ must consist of six relatively prime factors. In this category we form all possible combinations of primes with products less than $10^6/2$ and find seven quads below one million. The perimeters and the corresponding generators are:

P	(s, t)
317460	(286, 269), (330, 151), (370, 59), (390, 17)
543660	(390, 307), (410, 253), (442, 173), (510, 23)
554268	(374, 367), (418, 245), (442, 185), (494, 67)
570180	(390, 341), (430, 233), (442, 203), (510, 49)
570570	(385, 356), (399, 316), (429, 236), (455, 172)
949620	(490, 479), (510, 421), (570, 263), (646, 89)
986700	(506, 469), (550, 347), (598, 227), (650, 109)

Thus the primitive quad with the shortest common perimeter is (153868, 9435, 154157), (99660, 86099, 131701), (43660, 133419, 140381), (13260, 151811, 152389).

In the course of this investigation 175 primitive triplets with common perimeters less than one million were found. The set with the smallest common perimeter is:

P	a	b	c	s	t
14280	7080	119	7081	60	59
14280	5032	3255	5993	68	37
14280	168	7055	7057	84	1

The smallest non-primitive triplet has a perimeter of 120, see *School Science and Mathematics* **49**, 680, (1949). The shortest perimeters common to 1, 2, 3, 4, 5, 6, 7, 8, 10 Pythagorean triangles are listed in the *American Mathematical Monthly*, **56**, 632, (1949). Other treatments of the common perimeter problem have been given in the latter magazine, **56**, 32, 404, (1949).

Also solved partially by P. N. Nagara, College of Agriculture, Thailand, who found the set with common perimeter 554268.

The proposer also reports two sets of primitive Pythagorean triangles with equal perimeters and the perimeter a square number, namely: $1776873728 + 791591775 + 1945224353 = 243331328 + 2128246575 + 2142111953 = 4513689856 = (67184)^2$ and $7977955600 + 2160387999 + 8265292001 = 2845155600 + 7519095359 + 8039384641 = 18403635600 = (135660)^2$.

QUICKIES

From time to time as space permits this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 12. A cube of wood 3″ on each edge is to be cut into cubes 1″ on each edge. If, after each cut with a saw, the pieces may be piled in any desired manner before making the next cut, what is the smallest number of different "cuts through the pile" that will accomplish the desired dissection? [*Submitted by Frank Hawthorne*].

Q 13. Given a circle with radius *OA*, a perpendicular diameter *BOC* extended to *D*, and a tangent *DE*. Reflect *DE* onto *BD* without using compasses. [*Submitted by Fred Marer*].

Q 14. Consider the two series of integers,

$$A(2^n - 1):\ 1,\ 3,\ 7,\ 15,\ 31,\ \ldots,$$

$$B(2^n + 1):\ 3,\ 5,\ 9,\ 17,\ 33,\ \ldots\ .$$

Show that [*a*] The second integer (3) of *A* divides alternate integers of *B*, beginning with the first integer; [*b*] no other integer of *A* (other than 1) divides any integer of B; [c] every integer of B divides an infinite number of integers of A. [*Submitted by Dewey Duncan*].

Q 15. Find the sum of the squares of the coefficients in the expansion of $(a + b)^n$.

Q 16. Express sin $5x$ in terms of sin x. [*Submitted by Samuel Skolnik*].

Q 17. Two flights of bombers were flying at 300 m.p.h. on converging courses 30° apart, each flight being 240 miles from the rendezvous. From above each flight, a fighter plane, flying at 500 m.p.h., flew to the other bomber flight and returned, continuing the shuttle until the bomber flights met. One fighter always headed directly toward his objective, while the other fighter always flew an interception course. Which fighter flew the greater distance, and how much farther did he fly? [*Submitted by R. E. Horton*].

ANSWERS

A 17. Both fighters flew the same distance, 400 miles, since each flew for 240/300 hours during the operation.

A 16. By De Moivre's theorem $\cos 5x + i \sin 5x = (\cos x + i \sin x)^5 = \cos^5 x + 5i \cos^4 x \sin x - 10 \cos^3 x \sin^3 x - 10i \cos^2 x \sin^3 x + 5 \cos x \sin^4 x + i \sin^5 x$. Equating the imaginary parts, $\sin 5x = 5 \cos^4 x \sin x - 10 \cos^2 x \sin^3 x + \sin^5 x = 5(1 - \sin^2 x)^2 \sin x - 10(1 - \sin^2 x)\sin^3 x + \sin^5 x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$.

A 15. The possible ways in which n objects can be chosen from n red and n black objects is

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}.$$

But $\binom{n}{k} \equiv \binom{n}{n-k}$, so we have $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$.

A 14. Express the integers of A and B in dyadic form,

$$A: 1, 11, 111, 1111, 11111, \ldots,$$

$$B: 11, 101, 1001, 10001, 100001, \ldots .$$

Attempt to perform the divisions and the three propositions are immediately obvious.

A 13. Draw AE intersecting BD at F. Then $DE = DF$, since arc AB = arc AC, arc AB + arc CE = arc AC + arc CE, whence angles DFE and and FED are equal.

A 12. Six cuts; for no matter how the cutting is done, the 6 faces of the central cube must result from separate cuts. The job may be done without any piling at all.

MATHEMATICAL MISCELLANY

Edited by

Charles K. Robbins

Let us know (briefly) of unusual and successful programs put on by your Mathematics Club, of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the MATHEMATICS MAGAZINE or concerning other "matters mathematical" will be welcome. Address: CHARLES K. ROBBINS, Department of Mathematics, Purdue University, Lafayette, Indiana.

Letter from Benjamin Franklin, Esq; of Philadelphia, to Peter Collinson, Esq; at London

Sir, According to your request, I now send you the Arithmetical Curiousity, of which this is the history.

Being one day in the country, at the house of our common friend, the late learned Mr. *Logan*, he shewed me a folio *French* book, filled with magic squares, wrote, if I forget not, by one M. *Frenicle*, in which he said the author had discovered great ingenuity and dexterity in the management of numbers; and, though several other foreigners had distinguished themselves in the same way, he did not recollect that any *Englishman* had done any thing of the kind remarkable.

I said, it was perhaps, a mark of the good sense of our *English* mathematicians that they would not spend their time in things that were *difficiles nugae*, incapable of any useful applicatory. He answered, that many of the arithmetical or mathematical questions publickly proposed and answered in *England*, were equally trifling and useless. Perhaps the considering and answering such questions, I replied, may not be altogether useless, if it produces by practice an habitual readiness and exactness in mathematical disquisitions, which readiness may, on many occasions, be of real use. In the same way, says he, may the making of these squares be of use. I then confessed to him, that in my younger days, having once some leisure, (which I still think I might have employed more usefully) I had amused myself in making these kind of magic squares, and, at length, had acquired such a knack at it, that I could fill the cells of any magic square, of reasonable size, with a serie of numbers as fast I could write them, disposed in such a manner, as that the sums of every row, horizontal, perpendicular, or diagonal, should be equal; but not being satisfied with these, which I looked on as common and easy things, I had imposed on myself more difficult tasks, and succeeded in making other magic squares, with a variety of properties, and much more curious. He then shewed me several in the same book, of an uncommon and more curious kind; but as I thought none of them equal to some I remembered to have made, he desired me to let him see them; and accordingly, the next time I visited him, I carried him a square of 8, which I found among my old papers and which I will now give you, with an account of its properties.

The properties are,

1. That every strait row (horizontal or vertical) of 8 numbers added together, makes 260, and half each row half 260.

2. That the bent row of 8 numbers, ascending and descending diagonally, *viz.* from 16 ascending to 10, and from 23 descending to 17; and every one of its parallel bent rows of 8 numbers make 260. Also the bent row from 52, descending to 54, and from 43 ascending to 45; and every one of its parallel bent rows of 8 numbers, make 260. Also the bent row from 45 to 43 descending to the left, and from 23 to 17 descending to the right, and every one of its parallel bent rows of 8 numbers make 260. Also the bent row from 52 to 54 descending to the right, and from 10 to 16 descending to the left, and every one of its parallel bent rows of 8 numbers make 260. Also the parallel bent rows next to the above-mentioned, which are shortened to 3 numbers ascending, and 3 descending, etc. as from 53 to 4 ascending, and from 29 to 44 descending, make, with the 2 corner numbers 260. Also the 2 numbers 14, 61 ascending, and 36, 19 descending, with the 4 lower numbers situated like them, *viz.* 50, 1 descending and 32, 47 ascending, make 260. And, lastly, the 4 corner numbers, with the 4 middle numbers, make 260.

So this magical square seems perfect in its kind. But these are not all its properties; there are 5 other curious ones, which, at some other time, I will explain to you.

Mr. *Logan* then shewed me an old arithmetical book, in quarto, wrote, I think, by one *Stifelius*, which contained a square of 16, that he said he should imagine must have been a work of great labour; but if I forget not, it had only the common properties of making the same sum, *viz.* 2056, in every row, horizontal, vertical, and diagonal. Not willing to be out-done by Mr. *Stifelius* even in the size of my square, I went home, and made, that evening, the following magical square of 16, which besides having all the properties of the foregoing square of 8, i.e. it would make the 2056 in all the same rows and diagonals, had this added, that a four square hole being cut in a piece of paper of such a size as to take in and shew through it, just 16 of the little squares, when laid on the greater square, the sum of the 16 numbers appearing through the hole, wherever it was placed on the greater square, should likewise make 2056. This I sent to our friend the next morning, who, after some days sent it back in a letter, with these words: ''I return to thee thy astonishing or most stupendous piece of the magical square, in which'' - but the compliment is too extravagant, and therefore, for his sake, as well as my own, I ought not to repeat it. Nor is it necessary; for I make no question but you will readily allow this square of 16 to be the most magically magical of any magic square ever made by any magician.

I did not, however, end with squares, but composed also a magick circle, consisting of 8 concentric circles, and 8 radial rows, filled with a series of numbers, from 12 to 75, inclusive, so disposed as that the numbers of each circle, or each radial row, being added to the central number 12, they make exactly 360, the number of degrees in a circle; and this circle has moreover, all the properties of the square of 8. If you desire it, I will send it; but at present, I believe, you have enough on this subject.

I am, etc.

B. F.

A Brief Derivation of the Euler Exponential Form

Robert C. Yates

Particular solutions of $y'' + y = 0$ are obviously

$$e^{ix}, \cos x, \sin x.$$

However, since a differential equation of second order can have but *two* linearly independent solutions, constants a_1, a_2, a_3, not all zero must exist such that

$$a_1 e^{ix} + a_2 \cos x + a_3 \sin x \equiv 0,$$

with derivative $i \cdot a_1 e^{ix} - a_2 \sin x + a_3 \cos x \equiv 0$.

For $x = 0$, these two relations produce

$$a_1 + a_2 = 0, \text{ and } i \cdot a_1 + a_3 = 0$$

and thus either becomes

$$e^{ix} \equiv \cos x + i \cdot \sin x.$$

U.S.M.A.

AN INVITATION

A short time ago a ''candid'' radio announcer asked three casually selected people how to divide 7/8 by 3/4. One of them replied ''You can't do it. We learned to divide integers but you can't divide fractions.'' The second said ''You multiply them.'' The third said ''You change 3/4 to .75 and multiply 7/8 by .75.'' Obviously the correct rule had little meaning for them.

Many are in similar dilemmas about points in elementary and or advanced mathematics. And no doubt there is always some one who could clarify any such point.

The usual class or text book explanation assumes familiarity with the entire sequence of mathematics leading up to the point in question. But it is a long road from arithmetic to topology, say. However there are men, rare I suspect, who can short cut this trip by use of simple language and familiar illustrations. There is great need of such discussions by people in almost all walks of life.

Choose your own concept any place in mathematics, elementary or advanced. The acceptableness of your article will depend upon how clearly it explains your topic for those who don't speak technical mathematics, and not at all upon the originality or profundity of your mathematics as such. Originality and versatility can enter generously in presentation.

On the other hand if you have points you desire to have discussed, send them in and we will try to have them elucidated.

Glenn James

OUR CONTRIBUTORS

Arthur Erdélyi, Professor of Mathematics, California Institute of Technology, was born in Budapest, Hungary in 1908. He received his education in Hungary and Czechoslovakia, holds the degrees of Dr. rer. nat. (Prague, '38) and D.Sc. (Edinburgh, '40); and is a Fellow of the Royal Society of Edinburgh.

He first came to the California Institute of Technology as a visiting professor in 1947 and returned, to take up his present position, in 1949. He is an analyst interested principally in the theory of special functions.

Marlow Sholander, Assistant Professor, Washington University, St. Louis was a Summerfield Scholar while attending the University of Kansas (A.B. '36, M.A.). He received the Ph.D. in 1949 from Brown University, where he had taught during the war years. His interests, other than mathematical, are dominated by an extraordinary fondness for bridge.

William R. Ransom, born in Chicago in 1876, received Master's degrees at Tufts College and at Harvard University. He taught mathematics and sometimes physics and astronomy at Tufts over a period of fifty years. Among the students in his classes later to become famous were Norbert Wiener and Vannevar Bush. He specialized in freshman and sophomore subjects and developed a number of textbooks in these subjects.

Joseph Milkman, Assistant Professor of Mathematics, United States Naval Academy, was born in Brooklyn, New York in 1912, and attended Brooklyn College (B.S. '34; M.A. '37). He taught in the New York high schools, in the Defense Training Institute and served as a physicist with the Remington-Rand Research Laboratory before joining the Annapolis faculty in 1946. At present he is working on his doctorate at New York University.

Herbert Busemann, Professor of Mathematics, University of Southern California, was born in 1905 in Berlin, Germany. He studied at Munich, Paris, Rome and Gottingen (Ph.D. '31) and became an assistant in mathematics at Gottingen, 1931-33 and a lecturer at Copenhagen 1933-36. After coming to this country in 1936, Dr. Busemann spent three years at the Institute for Advanced Study and then taught at Swarthmore, Johns Hopkins, Illinois Institute of Technology and Smith College before accepting his present appointment. Well known as a geometer, Professor Busemann is the author of the book, "Metric Methods in Finsler Spaces" and numerous research articles. At present he is working on a text in projective geometry.